MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 101 No. 4

MEETING OF 1941 APRIL 9

Professor S. Chapman, President, in the Chair

The election by the Council of the following Fellows was duly confirmed :—

Ronald Frank Newby, 79 St. John's Grove, Harrogate, Yorks; and
Chandrakant Gajanan Pendse, B.Sc., M.A., Ph.D., Colaba Observatory, Bombay, India.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Clements Markham Colbeck, 38 Glen Drive, Stoke Bishop, Bristol, 9 (proposed by R. L. Waterfield);
H. E. Krumm, Sonnenschein, Leeuwendal Crescent, Cape Town, S. Africa (proposed by J. Jackson); and
George Macfarlane Sisson, M.A., 4 Archbold Terrace, Jesmond, Newcastle-on-Tyne (proposed by Cyril Young).

Seventy-seven presents were announced as having been received since the last ordinary meeting, including :

F. Berthoud, Traité d'Horlogerie (1755) and Essai sur l'Horlogerie (1763) (presented by D. Carder-Davies).

The President announced that after the issue of the next number of *Occasional Notes* the publication of this periodical would be temporarily suspended (probably for the duration of the war); in its place the Council had arranged to pay a block subscription to *The Observatory*, which would be sent gratis to every Fellow in place of *Occasional Notes*.

The President further announced that the Council was anxious to make the Library more accessible to Fellows wishing to use it, and he called for volunteers to take charge and act as librarian on certain days to be arranged.

The President further called for volunteers to sleep on the premises at Burlington House, in a rota, in order to assist in fire-fighting if occasion should arise.

MEETING OF 1941 MAY 9

Professor S. Chapman, President, in the Chair

The election by the Council of the following Fellows was duly confirmed:—

Archibald Brown, M.A., Trinity College, Cambridge;

Cecil Charles Hilton Rowe, F.R.S.A., F.I.L., c/o The Bank of England, Whitchurch, Hants; and

Alan J. Skinner, 64 Cambridge Road, Anerley, London, S.E. 20.

The following candidate was proposed for election as a Fellow of the Society, the name of the proposer from personal knowledge being appended:—

Colonel Frederick William Nicholls, Staddle Barn Cottage, Coombe Bisset, Wilts (proposed by F. J. M. Stratton).

Twenty-nine presents were announced as having been received since the last meeting, including:

Royal Observatory, Greenwich: Catalogue of 20554 Faint Stars in Cape Astrographic Zones $-40°$ to $-52°$; and

F. W. Sohon, S.J., The Stereographic Projection (presented by the publishers).

The President referred to the death of Miss Annie Cannon. He said:

"The Society has to mourn the loss of its distinguished Honorary Member, Miss Annie J. Cannon, who, since 1915, when Lady Huggins died, has been our only Honorary Member. The category of Honorary Member is one that was invented in order that the Society might pay fitting recognition to the distinguished astronomical contributions from women, at a time when women were deemed to be excluded from participation in the Fellowship of the Society by the terms of the Society's Charter. Happily this pointless restriction is now removed. Miss Cannon was elected Honorary Member at an important stage in her great work of cataloguing and classifying stellar spectra, namely, when she had just completed this work for 150,000 stars. Before her retirement she had materially increased this great total. Her retirement from her official work by no means lessened her interest in astronomy, and many of us had the pleasure of meeting this gracious and charming personality at the Stockholm Conference of the International Astronomical Union. Let us stand for a few moments in silent tribute to her memory."

The President announced that, owing to the lack of sufficient volunteers for fire-watching and Library duties, the schemes announced at the last meeting would lapse.*

* *Editorial Note.*—Since the date of the May meeting it has proved necessary to start the scheme of fire-watching with a limited number of volunteers. An announcement to this effect has already been printed in *The Observatory*, **64**, 92 (June 1941). Any Fellow who can undertake such duties should inform the Assistant Secretary, from whom full particulars may be obtained.

IONISATION EQUILIBRIUM IN A CONVECTIVE REGION

Sir A. S. Eddington, O.M., F.R.S.

(Received 1941 April 2)

1. The layers in which hydrogen is near the middle stage of ionisation form a very abnormal region in a star. The specific heat is large and the adiabatic constant Γ is much below the usual value $\frac{5}{3}$, with the result that the material is convectively unstable. According to Unsöld's calculations *, the unstable region in the Sun has a thickness of 300 km., extending from temperature 6500° to 18000° and from pressure 7000 to 40000 dynes cm.$^{-2}$. These figures furnish only a rough guide, because they take no account of the considerable modification of the pressure and temperature distribution which will result from the convection currents themselves.

A particle falling freely under the Sun's gravity through a distance of 300 km. acquires a velocity of 13 km. per sec. This is evidently an extreme upper limit to the speed which the vertical currents can attain. As a working upper limit we might adopt 5 km. per sec. as a reasonable speed.

In certain circumstances the effect of convection is to substitute the adiabatic for the radiative pressure-temperature gradient. This, however, does not apply to the critical hydrogen region. Unless the currents are rapid enough to transport a substantial fraction of the energy which emerges as radiation from the star, radiative equilibrium applies. It is easily computed that currents limited to 13 km. per sec. will not transport enough heat to affect the radiative gradient seriously. Of the two equations used in the ordinary equilibrium theory, (1) the equation of radiative equilibrium and (2) the hydrostatic equation, the hydrostatic equation breaks down first. It ceases to apply when the speed of the currents approaches $\sqrt{(P/\rho)}$; at 13 km. per sec. the failure would be complete. The dynamical effect of the currents is therefore more important than their thermodynamical effect.

The purpose of this paper is to call attention to another consequence of convection which may well be the most important of all. *Ionisation equilibrium is reached rather slowly, and for quite small speeds of convection the state of ionisation will fail to keep pace with the changing pressure and temperature of the material.*

We shall first make a quantitative estimate of the time lag.

2. By Planck's law we can calculate the number of photons n, with energy greater than 13·53 electron-volts and therefore capable of ionising hydrogen, contained in a cu. cm. of radiation at a given temperature T. Then, in equilibrium, $\frac{1}{4}nc$ ionising photons per sq. cm. will pass outwards across the level T. If each of them ionises one hydrogen atom, a mass $w = \frac{1}{4}nc\mathrm{H}$ of hydrogen is ionised per second above the level T, H being the mass of a hydrogen atom. The decimal logarithms of n and $\frac{1}{4}nc$, and the mass w in grams, are given in the following table:—

T	log n	log $\frac{1}{4}nc$	w
24000	12·992	22·866	·123
18000	11·901	21·775	·0100
15000	11·050	20·925	·00141
12000	9·808	19·683	·0000805

Consider a column of descending material between the levels $T_1 = 15000°$ and $T_2 = 9000°$. These levels are chosen so that in equilibrium conditions nine-tenths of

* *Physik der Sternatmosphären.* The data stated here and in § 2 are based on Table 36, p. 144, used in conjunction with the graphs in fig. 125, p. 382.

the ionisation of hydrogen occurs between them. (Taking the corresponding electron pressures from Unsöld's calculation, and applying the ionisation formula for hydrogen, we obtain $x_1 = 0{\cdot}94, x_2 = 0{\cdot}04$.) The pressures at these levels (from Unsöld) are $P_1 = 35000$, $P_2 = 15000$. The weight of the column per sq. cm. is the pressure-difference 20000 dynes cm.$^{-2}$, and the corresponding mass is 0·73 gm. Taking half the mass to be hydrogen, the column contains 0·36 gm. of hydrogen. Its height is 150 km.

Following the ideas introduced by Zanstra in his theory of planetary nebulæ, we regard the ionisation of the hydrogen in the column as being produced (directly or indirectly) by the photons entering at the base of the column. With the value of w computed above, namely ·00141, it would take 250 seconds to ionise the 0·36 gm. of hydrogen, or 230 seconds to perform the nine-tenths ionisation under consideration. Hence, if the average speed of the current exceeds $150/230 = 0{\cdot}65$ km. per sec., the material descends faster than the radiation can ionise it. We shall call the speed 0·65 km. per sec. calculated in this way the "standard speed."

The principle of the calculation is that, whether the material is in thermodynamical equilibrium or not, the purely internal recombinations and ionisations will approximately balance, each recombination involving the emission of an ionising photon which will produce a compensating ionisation when it is absorbed. Thus an increase in the degree of ionisation of the material can only be produced by a difference between the number of competent photons entering and leaving, and cannot exceed the number of entering photons $\frac{1}{4}nc$. If, for example, the current has five times the standard speed, the material will arrive at the level T_1 approximately one-fifth ionised. This will reduce the amount of recombination to $\frac{1}{25}$ of the equilibrium value, with a corresponding reduction in the emission of ionising photons. Thus the number of ionising photons leaving the column at its base is only $\frac{1}{25}$ of the number entering, and this source of loss can be neglected.

The calculation assumes that the radiation below the level T_1 has equilibrium composition; but actually it would be very seriously depleted of ionising photons by the un-ionised hydrogen which has fallen through the level T_1. The effect is therefore much greater than our calculation represents. Having regard to this depletion, a speed of 1 km. per sec. would probably be sufficient to bring the hydrogen to the 15000° level mostly un-ionised.

In equilibrium conditions a substantial part of the ionisation occurs from an excited level, requiring photons of lower energy. But this part of the ionisation is eliminated by convection at even slower speeds. By the same argument, an increase in the number of excited atoms can only be produced by a difference between the number of exciting photons entering and leaving the base of the column, and cannot exceed the number entering. This number is small compared with the number of ionising photons, since the energies are restricted to small ranges starting at 10·15 electron volts. Since there will be only one extra ionisation for each extra excitation*, this addition to the ionisation is negligible.

Similar delay occurs in an ascending column. The net rate of recombination † of the ions will not keep pace with the changing temperature and pressure, and the hydrogen will still be mostly ionised when it reaches the 9000° level.

It appears then that ionisation equilibrium will break down altogether when the convection is still too slow to affect appreciably the hydrostatic equation or the equation of radiative equilibrium. This break-down should therefore be the first effect of convection to be taken into consideration. It is accompanied by an even more radical break-down of excitation equilibrium.

3. The theory will perhaps become clearer if we trace the progress of a quantity of

* The excitation will pass from atom to atom until the ionisation occurs.

† A *net* recombination implies that the emitted photon escapes from the column without causing an ionisation. The net rate of recombination is therefore slower than the ordinary recombination rate.

gas round a circuit. The speed being limited to about 1 km. per sec., which is well under the velocity of sound, it may be assumed that the gas is in pressure equilibrium continually. We begin with a mass near the photosphere, un-ionised and starting to descend. The increasing compression causes the temperature to rise; and if the rise makes the mass hotter than the surrounding matter at the level reached, it will lose heat energy to its surroundings, and so dissipate its circulation energy. The ordinary argument is that, owing to the critical state of the hydrogen, the work done in compression is mainly used up in ionisation; so that the rise of temperature is abnormally small, and the mass gains heat from the surroundings instead of losing it. We have seen that this is not the whole truth. The first effect of compression is to raise the temperature (measured by the kinetic energy of the particles) to the normal extent; and the process of converting the extra heat energy into ionisation energy is slow. It would be very much slower were it not for the stellar radiation flowing through the mass which acts as a catalyst. Without making any net addition to the energy, it provides high-frequency radiation in return for low-frequency radiation; and our calculation of the rate of conversion has been based on the quantity of high-frequency radiation available for this exchange. The descent must be slow enough to allow time for the conversion into ionisation energy of a quantity of heat energy sufficient to prevent the temperature rising above that of the surroundings; otherwise the energy of circulation will be dissipated. But the rest of the conversion is deferred; and most of the hydrogen will be still un-ionised when it reaches the level 15000° at which, according to the equilibrium theory, ionisation would have been practically complete.

The descent will continue below the 15000° level until considerably higher temperatures are reached, furnishing an abundance of high-frequency radiation which rapidly finishes off the deferred ionisation. The fully ionised hydrogen is then sucked into an ascending current. Here the expansion due to the decreasing pressure causes the temperature to fall. The fall of temperature is checked by the heat released in recombination of the hydrogen ions. Here again the ordinary argument exaggerates the check by assuming the recombination to be instantaneous. The current must adjust itself to a speed which allows sufficient recombination to prevent the temperature falling below that of the surroundings; but most of the hydrogen is carried up uncombined. It remains in the layers at the top of the ascent until recombination is complete, and then starts off on another round.

In this outline we have for simplicity applied the condition for zero dissipation to every part of the circuit. Actually it is an integral condition which need only be satisfied by the circuit as a whole.

4. The break-down of ionisation equilibrium has so many consequences that it is difficult to know where to begin the discussion. One effect is to extend considerably the original zone of convection, since the ionisation and de-ionisation of the hydrogen is not completed at the levels computed from the equilibrium theory. Taking a mean of the descending and ascending columns, the ionisation gradient covers a wider region and is correspondingly less steep. There is a limit to the flattening of the gradient, since it must supply the motive power maintaining the currents. Presumably the actual speed of circulation is determined by the condition that it increases until the ionisation gradient is reduced to a value only just sufficient to maintain convective instability.

It is necessary to verify that the large deviation from ionisation equilibrium described in § 2 is not inconsistent with a gradient sufficient to maintain the circulation. The ordinary calculation of the "ratio of specific heats" Γ assumes equilibrium ionisation; the actual Γ will be a function of the speed of the current. Consider a quantity of hydrogen transported from the 9000° to the 15000° level at a speed which makes $\Gamma = \frac{4}{3}$, this being the value which in typical conditions gives convective instability. For this value of Γ the increase of ionisation energy is equal to the increase of translatory energy of the atoms; so that the required ionisation energy is equal to the translatory energy

corresponding to 6000°. But the ionisation energy of a hydrogen atom is equal to its translatory energy at 105000°. Thus the speed must be such that $\frac{1}{17}$ of the hydrogen is ionised when the 15000° level is reached.*

I will consider especially the effect on the coefficient of opacity, because, if I am right, it makes the deviation from ionisation equilibrium *regenerative*; that is to say, it modifies the pressure-temperature distribution in such a way as to cause even larger deviations than those computed from the unmodified distribution.

The radiation with energy below and above 13·53 electron-volts will be distinguished as "radiation A" and "radiation B" respectively. We notice first that the continuous absorption of radiation A by the hydrogen will be abnormally small both in the ascending and descending columns. It depends on ionisation from an excited level, and disappears because the hydrogen in the ascending column is mainly ionised and the hydrogen in the descending column is mainly in the ground state, excitation being eliminated even more thoroughly than ionisation (§ 2). Absorption by oxygen is abnormally low for the same reason.

In the descending column, absorption by metallic atoms with low ionisation and excitation potential will be reduced, but not so drastically. The reduction will be roughly in proportion to the reduced ionisation of the hydrogen, because the rapidity of functioning of the photo-electric mechanism of absorption is proportional to the number of free electrons available for capture. Free-electron scattering is also reduced proportionately to the hydrogen ionisation. Radiation B will be strongly absorbed by the neutral hydrogen; but most of it is eliminated before entering the column, and in any case a strong absorption at one extremity of the Planck curve has little effect on the Rosseland-mean opacity.

The opacity of the ascending column to radiation A will probably not be much changed. Since the hydrogen is almost fully ionised instead of (on the average) half ionised, the metallic absorption will be doubled; but the hydrogen and oxygen absorption will be eliminated. The ascending column is highly transparent to radiation B, and a considerable amount will get through to the uppermost levels, where it will be stepped down to lower frequencies.

Taking the descending and ascending columns together the net result will be a substantial increase of transmission. The increase will be very large if, in static conditions, hydrogen and oxygen furnish the major part of the absorption; but it seems more likely that metallic absorption preponderates, and we shall not count on more than a moderate reduction of opacity.† In combining the two columns the partial segregation of opacity to high-frequency and low-frequency radiation leads to increased transmission, just as two windows, one of red and the other of green glass, transmit more light than two windows each consisting of half-thicknesses of red and green glass superposed. By the equation of radiative equilibrium a reduced opacity gives a larger pressure for the same temperature or optical depth.

For definiteness suppose that the specific opacity κ is reduced in the ratio $\frac{1}{9}$. Considering a fixed optical depth τ, the temperature is unaltered, and therefore (approximately) P/ρ is unaltered. Since $dP = -g\rho dx$, dx is approximately unaltered, and the height of the column between 15000° and 9000° remains about 150 km. The actual opacity k (assumed to be mainly metallic absorption ‡) is proportional to κP. It follows that P and ρ must be altered in the ratio $\kappa^{-\frac{1}{2}}$; so that k is altered in the ratio $\kappa^{\frac{1}{2}}$, and $d\tau = -k\rho dx$ is unchanged. Thus P is increased 3-fold; and the weight and mass of

* Allowing for the other elements and free electrons present, the fraction would be about $\frac{1}{15}$.

† Some idea of the relative contributions can be obtained from Unsöld, *Physik der Sternatmosphären*, p. 126, Table 29; but the oxygen absorption is there grouped with the metallic absorption. The assumed proportion of the elements is, of course, very uncertain.

‡ This is the most unfavourable assumption. If the opacity is mainly hydrogen and oxygen absorption, the pressure is increased 9-fold.

hydrogen in the column are increased 3-fold. This reduces the "standard speed" to 0·22 km. per sec.

On the other hand, the increased pressure at T_1 and T_2 reduces the equilibrium ionisation at these levels, so that the region of convective instability is shifted inwards to higher temperatures for which w is greater. This will to some extent counterbalance the reduction of the standard speed found in the last paragraph. But, either by reduction of the standard speed or by shifting the limits of the convective region, the result of the reduced opacity is to carry the deviation from ionisation equilibrium farther into the interior of the star, thereby increasing the mass of matter affected and enhancing its general importance.

I think therefore that the initial figures found or suggested in § 2 by no means exaggerate the deviation from ionisation equilibrium.

5. In an accompanying paper "On the Cause of Cepheid Pulsation," an argument which seems to me of considerable weight leads to the conclusion that the quantity of hydrogen in the critical region is from 10 to 100 times greater than the ordinary calculations allow. The present investigation arose out of this discrepancy. So far as I can see, the reduced opacity, together with the increased width of the critical region, will increase the quantity of hydrogen to the required amount.

Another consequence has a more general interest. We have seen that the ascending column is highly transparent to radiation B, so that an abnormal quantity of high-frequency radiation will penetrate to the outermost layers. Moreover, radiation B is emitted in the recombination of the hydrogen convected in an ionised state to the top of the column. For both these reasons the radiation in the photosphere and reversing layer will contain a proportion of high-frequency constituents greatly in excess of that given by Planck's law. This will introduce important modifications in the numerous calculations of ionisation, excitation, etc., associated with spectro-photometry; and it may be hoped that it will remove some of the existing discrepancies.

[*Added June* 10.] In the foregoing discussion it has been assumed that collision excitation and ionisation are negligible. I have not been able to find physical data (experimental or theoretical) adequate for a precise calculation of collision effects; but it appears that they may be more potent than I expected, and that the time-lag of ionisation equilibrium may be determined by the rate of collision transfer instead of photo-electric transfer. This may make the region where ionisation equilibrium breaks down less extensive; but it seems clear that even the maximum estimate of the rate of collision transfer will still leave effects of considerable astrophysical importance. I think that, for currents of 1 km. per sec., the break-down will at least cover the upper half of the convective layer in the Sun; and in the giant stars, where the temperatures and densities are lower, large deviations will continue. In particular, the excess of high-frequency radiation in the photosphere, though somewhat reduced, will remain important.

Summary

It is shown that the equilibrium calculation of ionisation and excitation breaks down altogether in the convective layer in the Sun. The descending currents convect neutral hydrogen to a depth where the equilibrium formula would give almost complete ionisation, and the ascending currents convect ionised hydrogen nearly to the photosphere. This appears to be much the most important consequence of convection since it occurs at current-speeds too slow to affect the dynamical and thermodynamical equations. The effect on the opacity is investigated. One result, which bears on a great many problems concerning the outer layers of a star, is that in the photosphere and reversing layer the proportion of high frequency radiation will be greatly in excess of that given by Planck's law.

ON THE CAUSE OF CEPHEID PULSATION

Sir A. S. Eddington, O.M., F.R.S.

(Received 1941 April 2)

1. *Introduction.*—The most outstanding difficulties in the pulsation theory of Cepheid variables are—

(1) The quarter-period retardation of phase between the flow of heat in the main part of the interior and the outflow from the surface.

(2) The existence of a period-luminosity relation.

We shall find reason to believe that there is a close connection between (1) and (2), and that both depend on—

(3) The existence of a layer not far below the photosphere where hydrogen is in the mid-stage of ionisation, so that it is ionised and de-ionised in the course of the pulsation.

The critical hydrogen layer is a comparatively new factor in the problem, since it could not be deemed important until the high abundance of hydrogen in the stars was recognised in 1932. It has been investigated by Unsöld and others in connection with surface phenomena on the Sun, but I think it has not hitherto been considered in connection with Cepheid pulsation. To show the general plausibility of connecting it with (1) and (2), we remark :

(*a*) It is difficult to see any possible form of explanation of the period-luminosity relation which does not make it depend on a critical ionisation. A period-luminosity relation is effectively a density-mass relation, since the period is sensitive to changes of density and the luminosity to changes of mass. It therefore implies that, for any given mass, there is only a narrow range of density in which the star is pulsatorily unstable. The *transience* of the instability—the fact that it disappears both at slightly higher and slightly lower densities—greatly limits the field in which we can seek for an explanation. Presumably we must look for some equally transient modification of the properties of stellar material. The only kind of transient modification that is known is the sharp drop in the ratio of specific heats Γ, when the predominant element or elements are at the mid-stage of an ionisation. I considered this possibility in an early investigation *; but, following the ideas of the time, I took the predominant element to be a heavy element, such as iron, undergoing the L-ionisation. This ionisation occurs in the deep interior, and in a giant star depends mainly on the central temperature. The hypothesis that pulsatory instability occurs when the heavy elements in the interior are near the middle of the L-ionisation therefore required that the Cepheids should all have nearly the same central temperature—which is far from true. The substitution of hydrogen as the predominant element reopens the discussion, since it shifts attention from the deep interior to the layer (3) near the surface, which is now the only region in which there can be a serious drop of Γ.

(*b*) The great difficulty of accounting for the phase retardation of roughly a quarter-period can only be appreciated by a close study of the equations. In a previous investigation †, I divided the star into three regions: A, the interior region with temperature above 100,000°, where the adiabatic approximation is nearly perfect; C, the exterior region with temperature below 40,000°, with negligible capacity for storing heat; B, the intermediate region. There could be no appreciable change of phase in A and C; and a detailed study of the differential equation applicable to region B also indicated no

* *Internal Constitution of the Stars*, p. 204.

† *M.N.*, 87, 539, 1927.

change of phase.* One way—apparently the only way—of avoiding this *impasse* is by some means to add heat-storage capacity to region C. Such extra heat-storage is provided by the ionisation and de-ionisation of the critical hydrogen layer which lies in region C.

(*c*) A crucial question is whether the amount of hydrogen in the critical layer is sufficient to meet the demands of our theory. It has to store (in the form of ionisation energy) the temporary excess of radiant energy accumulating during the quarter-period delay in emission. The rough calculations are fairly encouraging, showing that the order of magnitude is not far wrong. When greater precision is sought, we encounter the difficulty that the critical layer in Cepheids corresponds to temperatures chiefly between 10,000° and 20,000°. This is a region where our knowledge is especially uncertain; it is out of reach of direct observation, and the simplifications adopted at higher temperatures are inapplicable or dubious; moreover, it is convectively unstable. The somewhat conjectural calculations already existing give too little hydrogen for our purpose. But we might argue that the estimate obtained from the Cepheid theory is the better founded, and may justifiably be used to extend our knowledge of this difficult region. The need for further light on this question has led me to a study of the effects of convection, contained in an accompanying paper.

It is generally agreed that the maintaining energy of the pulsation comes from the variation with temperature and density of the rate of liberation of sub-atomic energy. Through this variation heat is added to the material when it is at a high temperature and subtracted when it is at a low temperature, so that mechanical work is produced as in a thermodynamic engine. Pulsation occurs if this maintaining energy exceeds the dissipation. We have therefore to inquire whether the transient cause of Cepheid pulsation is a sudden increase of maintaining energy (per unit amplitude) or a sudden decrease of dissipation. All suggested forms of liberation of sub-atomic energy are extremely sensitive to temperature; so that if pulsation were associated with a transient peculiarity of the maintaining energy it would almost certainly occur at an approximately fixed central temperature. For example, an increase of maintaining energy, confined to a narrow range of density, would occur if there were an especially temperature-sensitive source of sub-atomic energy which came into play when the star had contracted so as to reach an appropriate central temperature, further contraction being arrested until this particular source was exhausted.† But this would not give the existing sequence of Cepheid variables, in which the stars of short period have much higher internal temperatures than those of long period.

We conclude therefore that *Cepheid pulsation is associated with a transient decrease of dissipation.*

There is therefore a suggested triple connection between (1) the sharp decrease of dissipation which permits pulsation, (2) the low value of Γ in the critical hydrogen layer, and (3) the quarter-period retardation of phase. In §§ 2–4 two links of this connection are tightened by obtaining an expression for the dissipation which shows explicitly that it depends on the phase-retardation and on the value of Γ in the outer layers. This part of the investigation is believed to be rigorous. The third link—the dependence of the phase-retardation on the value of Γ in the critical layer—does not admit of precise treatment, and is studied in § 5 in an over-simplified model.

In earlier investigations it was expected that the 90° phase-retardation would be revealed as a *general* property of the solution of the equations of oscillation. No such

* There exist solutions of the differential equation for region A which give change of phase (discussed by M. Schwarzschild); but the difficulty is that they require boundary conditions which give a larger change of phase in region B, and this in turn requires a still larger change of phase in region C incompatible with its low capacity for heat storage. (See also § 10.)

† It seems rather likely that such sources exist; but they explain the concentration of the Cepheids near particular points on the period-luminosity curve—not the curve itself (Gamow, *Birth and Death of the Sun*, p. 150).

property was found; and the present theory takes a different form. If a star is set in oscillation by a disturbance, the phase-retardation may be anything from 0° to 180°, or more, according to the extent of the region in which the value of Γ is depressed. But there is an optimum extension of the region, with a corresponding phase-retardation, which makes the dissipation a minimum; and when, in the course of evolution of a star, the region reaches this extension, the disturbance, instead of decaying, swells into Cepheid pulsation. Thus the retardation in Cepheids is dependent on the condition for minimum dissipation. It is not likely to be exactly 90° (the mean observational value is about 70°); and it may depend to some extent on the period.

In the main the case for the present theory is that we seem able to eliminate all alternatives. There are, however, two results which give it positive support. In § 6 we find that, if the criterion for pulsatory instability is concerned with the state of ionisation of the outer layers, a very general application of the principle of homology yields a fairly correct period-luminosity relation. In § 9 we find that the theory gives an upper limit of about $0^m.8$ (bolometric) to the magnitude range, which is in satisfactory agreement with observation.

2. *Calculation of the Dissipation.*—Let F be the outward flow of radiation per second across a sphere of radius ξ, and let m be the mass inside the sphere. The gain of heat in an element at ξ due to the transfer of radiation is $-dF/dm$ per gm. per sec. To this must be added the heat ϵ supplied by liberation of sub-atomic energy. The gain of heat per gm. per sec. is accordingly

$$\frac{dQ}{dt} = \epsilon - \frac{dF}{dm}. \tag{2.1}$$

We fix attention on a gram of matter moving with the pulsation, so that m remains constant. We write

$$\xi = \xi_0(1+\xi_1), \qquad T = T_0(1+T_1), \qquad F = F_0(1+F_1), \qquad \epsilon = \epsilon_0(1+\epsilon_1),$$

etc., where ξ_0, T_0, . . . are the equilibrium values. Since $\epsilon_0 = dF_0/dm$, (2.1) becomes

$$\frac{dQ}{dt} = \epsilon_0\epsilon_1 - \frac{d(F_0F_1)}{dm}. \tag{2.2}$$

If a steady pulsation is maintained, the energy in the element is the same at the beginning and end of a cycle. Hence, during a complete cycle, the work done on the element by surrounding pressures is

$$W = -\int_0 \frac{dQ}{dt}dt. \tag{2.31}$$

The entropy of the element is also the same after a complete cycle; hence by the second law of thermodynamics

$$0 = \int_0 \frac{1}{T}\frac{dQ}{dt}dt. \tag{2.32}$$

Since $1/T = 1/T_0(1+T_1) = 1/T_0 - T_1/T_0$ to the first order in the amplitude, (2.32) gives

$$0 = \frac{1}{T_0}\int_0 \frac{dQ}{dt}dt - \frac{1}{T_0}\int_0 T_1\frac{dQ}{dt}dt$$

correct to the second order in the amplitudes. Hence by (2.31)

$$W = -\int_0 T_1\frac{dQ}{dt}dt \tag{2.33}$$

correct to the second order.

The advantage of using (2.33) rather than (2.31) is that in order to obtain W (which is a second order quantity) we need only calculate dQ/dt to the first order, whereas (2.31) requires a long investigation of the second order terms in dQ/dt.*

The quantities T_1, F_1, ϵ_1, etc., contain time factors $\cos(nt+\alpha)$ with various phases α. We shall denote by $(X)_Y$ the component of a periodic quantity X in the phase of the periodic quantity Y, and by $(X)_{iY}$ the component in the phase 90° greater. With this notation the time-factors are dropped, so that T_1, F_1, etc., are the semi-amplitudes. Then (2.33) gives for the average rate of doing work on a gram of matter

$$\frac{dW}{dt} = -\tfrac{1}{2}T_1\left(\frac{dQ}{dt}\right)_T$$

$$= -\tfrac{1}{2}T_1\left(\epsilon_0\epsilon_1 - \frac{d(F_0F_1)}{dm}\right)_T. \tag{2.4}$$

The two terms on the right represent respectively the maintaining energy (with reversed sign) and the dissipation. Hence for the whole star the maintaining energy E and the dissipation D are (per second)

$$E = \tfrac{1}{2}\int T_1\epsilon_0(\epsilon_1)_T dm, \tag{2.51}$$

$$D = \tfrac{1}{2}\int T_1\left(\frac{d(F_0F_1)}{dm}\right)_T dm. \tag{2.52}$$

Another form of (2.52) is useful. Let the *adiabatic* variation of temperature with pressure be given by

$$T \propto P^\theta \tag{2.6}$$

so that in adiabatic conditions $T_1 = \theta P_1$. In general conditions, we separate T_1 into two parts, namely $T_1' = \theta P_1$ resulting directly from the change of pressure, and $T_1'' = T_1 - \theta P_1$ resulting from the accession of heat Q. Since T_1'' has the same phase as Q, it differs 90° in phase from dQ/dt, and therefore contributes nothing to (2.33). We may therefore replace T_1 by T_1' or θP_1 in the subsequent steps, and (2.52) then becomes

$$D = \tfrac{1}{2}\int \theta P_1\left(\frac{d(F_0F_1)}{dm}\right)_P dm. \tag{2.7}$$

3. *Approximation in the Non-adiabatic Region.*—Throughout most of the interior the adiabatic approximation is nearly perfect, and we may take $\frac{1}{20}$ of the star's radius as a generous estimate of the thickness of the region in which it is inadequate. In parts of this region T_1, F_1, ρ_1 may be altogether different from the adiabatically computed values; but we shall show that the deviations of ξ_1, P_1 are on a much smaller scale.†

Let $\Delta\rho_1$ be the deviation of ρ_1 from the adiabatically computed value. If $\Delta\rho_1$ were constant throughout the non-adiabatic region, its volume would be changed in the ratio $(1-\Delta\rho_1)$ and its thickness in nearly the same ratio. This would give a modification $\Delta\xi_1$ of ξ_1 ranging from 0 at the inner boundary of the non-adiabatic region to $-\frac{1}{20}\Delta\rho_1$ at the surface. Allowing for the actual variation of $\Delta\rho_1$ in the region, the *maximum* value of $\Delta\xi_1$ is not greater than $\frac{1}{20}$ of the *mean* value of $\Delta\rho_1$; and a homogeneous comparison of mean with mean, or maximum with maximum, would give a ratio considerably less than $\frac{1}{20}$.

* The short method was used in the author's investigations (*M.N.*, **79**, 177, 1919; *I.C.S.*, § 134). Later writers seem to have overlooked this simplification, and have unnecessarily made the direct calculation of the second order terms in dQ/dt.

† The approximation which consists in neglecting $\Delta\xi_1$, ΔP_1 in comparison with $\Delta\rho_1$, ΔT_1, ΔF_1 was first used in treating the outer layers of a Cepheid in *M.N.*, **87**, 542, 1927.

By the rigorous formula (*I.C.S.*, equation (131.1))

$$\frac{d(P_0P_1)}{dP_0} = -(4+n^2\xi_0/g_0)\xi_1 \tag{3.1}$$

we have

$$P_1 = -\frac{1}{P_0}\int_0^{P_0}(4+n^2\xi_0/g_0)\xi_1 dP_0. \tag{3.2}$$

The factor $(4+n^2\xi_0/g_0)$ is about 5·5. It is nearly the same in all Cepheids, and changes very little in the non-adiabatic region. Thus P_1 is approximately $-5{\cdot}5\bar{\xi}_1$, where $\bar{\xi}_1$ is the mean value (averaged for equal steps of P_0) outside the point considered; and for the deviation ΔP_1 from the adiabatic value P_1

$$\Delta P_1/P_1 = \Delta\bar{\xi}_1/\bar{\xi}_1. \tag{3.3}$$

The proportionate deviation of P_1 will therefore not exceed that of ξ_1, and (owing to the averaging) it will generally be considerably less.

It is therefore a consistent approximation in the non-adiabatic region to neglect ΔP_1 and $\Delta\xi_1$ whilst retaining $\Delta\rho_1$, ΔT_1 and ΔF_1. The order of magnitude of ΔT_1 and $\Delta\rho_1$ is the same; in fact, ΔP_1 being negligible, the perfect-gas law gives $\Delta T_1 = -\Delta\rho_1$.*

We may therefore accept the values of ξ_1 and P_1 given by the usual equations of adiabatic pulsation as approximately valid up to the surface, the deviation in the non-adiabatic region being confined to density, temperature, flux of heat, and quantities derived from them. The error of P_1 can scarcely exceed 2 or 3 per cent. Since the adiabatic solution gives a constant phase for P_1 and $-\xi_1$ in all parts of the star, we shall distinguish this constant phase by the suffix $_C$. The dissipation formula (2.7) accordingly becomes

$$D = \tfrac{1}{2}\int\theta P_1\left\langle\frac{d(F_0F_1)}{dm}\right\rangle_C dm \tag{3.4}$$

$$= -\tfrac{1}{2}\int\frac{d(\theta P_1)}{dm}F_0(F_1)_C dm + \tfrac{1}{2}[\theta P_1F_0(F_1)_C] \tag{3.5}$$

by integration by parts. Note that the integration by parts is only possible with a phase-suffix $_C$ which represents *constant* phase; for example, it cannot be applied to (2.52).

4. *The Retardation of Phase.*—Consider a pulsating star in which—

(*a*) The index θ is constant throughout.

(*b*) Maximum radiation occurs a quarter period after minimum radius.

We shall show that D is negative.

Condition (*b*) asserts that at the surface F_1 differs 90° in phase from $-\xi_1$, so that $(F_1)_C = 0$. Hence the second term in (3.5) vanishes, and

$$-D = \tfrac{1}{2}\int\frac{d(\theta P_1)}{dm}F_0(F_1)_C dm. \tag{4.1}$$

The solutions of the equation of adiabatic pulsation show that P_1 increases continually from the centre outwards. According to our approximation (§ 3) this continues to apply in the non-adiabatic region. Hence, θ being constant, $d(\theta P_1)/dm$ is positive throughout.

It is also found that $(F_1)_C$ is positive in the adiabatic region; that is to say, the flow of heat is greatest at the time of greatest contraction.† Although this result is not estab-

* In the critical hydrogen layer the relation $\Delta T_1 = -\Delta\rho_1$ is inaccurate since the change of molecular weight through ionisation is appreciable.

† See, for example, the table in *I.C.S.*, p. 197; the quantity tabulated as F_1 is $-(F_1)_C$. The table on p. 202 illustrates a reversal of sign if there is a deviation from the usual law of opacity; but the value $\gamma' = 1{\cdot}355$ is now known to be much too small, and the region where the sign is reversed could not be so extensive as in the table.

lished with entire generality, being to some extent dependent on the adopted stellar model and the approximate law of opacity, the possible loopholes are, I think, not such as we should be likely to accept. In any case the region where $(F_1)_c$ might in certain circumstances have reversed sign is limited to the central part of the star, and the negative values are outweighed in the integral in (4.1) by large positive values farther out. We shall therefore disregard this contingency. In the non-adiabatic region $(F_1)_c$ falls from its positive value at the junction to its surface value zero.

Thus the elements of the integral in (4.1) are positive, and D is then negative.

The negative D means that pulsation will occur spontaneously even if no maintaining energy is provided by sub-atomic sources. If the conditions (*a*) and (*b*) hold in actual Cepheids, it follows that these stars maintain their pulsation by negative dissipation and not by sub-atomic stimulation—contrary to the general belief. Conversely, if negative dissipation is inadmissible, it follows that the conditions (*a*) and (*b*) are incompatible; in other words, the observed phase retardation is only possible because θ is not constant throughout the star.

The hypothesis of negative dissipation is not likely to be advocated except as a last resource. It is not necessarily contrary to thermodynamic principles; because the *steady* supply of energy ϵ_0 at high temperature in the interior could be converted into mechanical energy of pulsation if the constitution of the star provided a suitable "valve mechanism." * But, unless the conditions are widely different from what we suppose, there appears to be no mechanism which would regulate the flow of heat in the way required. Our result gives no new support to the hypothesis; it only shows that, in arbitrarily prescribing conditions (*a*) and (*b*), we have implicitly introduced it.

We shall assume that the hypothesis of negative dissipation can be ruled out. Our theorem then becomes a proof, by *reductio ad absurdum*, that the conditions (*a*) and (*b*) are incompatible. By (2.6), θ is connected with the adiabatic constant Γ. (If radiation pressure and variation of molecular weight are neglected, $\theta=(\Gamma-1)/\Gamma$.) Our conclusion is therefore that the quarter-period retardation of phase is impossible without a change of Γ in some part of the star. As explained in § 1, the great abundance of hydrogen makes the critical hydrogen layer the only region in which Γ can deviate seriously from its normal value $\frac{5}{3}$.

Arguing directly from (2.6), a low value of θ means that an adiabatic compression produces a relatively small increase of temperature; and therefore a given increase of temperature requires more than the normal amount of energy. In Cepheid conditions the only employment for the extra energy is in producing ionisation. The amount that can be so employed is small compared with the heat energy except in a region where the hydrogen is being ionised.

The present conclusion that phase-retardation cannot occur unless there is a region in which θ is abnormally small agrees with the investigation referred to in § 1 (*b*). The differential equations were solved for the case in which θ is constant (or a slowly varying parameter), and the phase-retardation was found to be insignificant. This negative result is now seen to be due to the fact that the condition essential for phase-retardation was left out, namely an abnormal drop of θ, or equivalently a region with abnormal capacity for storing heat. The study of the differential equation brings out the further point that, in order to be effective, the drop of θ must occur in the non-adiabatic region. It is therefore due to an ionisation which takes place at a temperature below 100,000°. This is a further proof, if any were needed, that the ionisation to be considered is that of hydrogen.

To show how negative dissipation can be avoided by admitting a low value of θ in the non-adiabatic region, we return to (3.4) and calculate separately the dissipations D_a, D_n in the adiabatic and non-adiabatic regions. Values at the junction of the two

* The "maintaining energy" of the pulsation, which depends on the *temperature-sensitiveness* of ϵ, must not be confused with the energy which maintains the star's heat, *i.e.* ϵ itself.

regions will be indicated by an accent, and surface values by an asterisk. We consider a star with quarter-period phase-retardation (so that $(F_1^*)_c = 0$), and with different constant values θ_a, θ_n of θ in the two regions.

In the adiabatic region $P_1 < P_1'$, and $d(F_0F_1)/dm$ is positive. Hence by (3.4)

$$0 < D_a < \tfrac{1}{2}\theta_a P_1' F_0' (F_1')_c. \tag{4.2}$$

It is easily seen that D_a is *substantially* less than the upper limit given.

Since there is little change of P_1 in the shallow non-adiabatic layer, we have approximately

$$D_n = \tfrac{1}{2}\theta_n P_1' F_0' \{(F_1^*)_c - (F_1')_c\} \tag{4.3}$$

$$= -\tfrac{1}{2}\theta_n P_1' F_0' (F_1')_c \tag{4.4}$$

since $(F_1^*)_c = 0$. By (4.2) and (4.4), $D_a + D_n$ will be negative unless θ_n is substantially less than θ_a.

This concludes the first stage of our investigation. It seems an almost inescapable conclusion that the critical hydrogen layer is the key factor on which the phase-retardation, the pulsatory instability and the period-luminosity relation depend. I therefore approach the remaining problems with a rather strong conviction that no other solution is possible, and that there must be a way out of the various difficulties which may appear when the details of the solution are considered.

5. *Phase-Retardation in the Critical Layer.*—If F_1 travels through the non-adiabatic region without loss of amplitude, we have

$$(F_1^*)_c = F_1' \cos\phi,$$

where ϕ is the phase-retardation. Hence by (4.3)

$$-D_n = \tfrac{1}{2}P_1' F_0' F_1' \theta_n (1 - \cos\phi). \tag{5.1}$$

Thus the condition for minimum dissipation is

$$\theta_n(1 - \cos\phi) \text{ is a maximum,} \tag{5.2}$$

and according to the foregoing theory pulsation will occur only in the neighbourhood of this maximum. We could therefore determine the phase-retardation ϕ in Cepheids from (5.2) if we knew the law of dependence of ϕ on θ_n. This dependence is the third leg of our three-cornered relation. We can scarcely hope to obtain a definite formula, but we shall discuss the main features of the problem in a simplified model.

We suppose the star in adiabatic pulsation to be surrounded by a non-adiabatic "blanket" in which θ is very small. By § 3, P_1 in the blanket is not abnormal, and therefore the temperature oscillation θP_1, produced directly by the expansion and contraction, is very small. The main temperature oscillation in the blanket is produced by the difference between the influx F_0F_1' and efflux $F_0F_1^*$ of radiation. Let s be the specific heat, and set

$$z = -\int s T_0 dm. \tag{5.31}$$

Then in the blanket

$$-d(F_0F_1) = sdm \cdot \frac{d}{dt}(T_0T_1),$$

so that, if T_1 varies as e^{int},

$$\frac{dF_1}{dz} = \frac{1}{F_0}\frac{dT_1}{dt} = \frac{in}{F_0}T_1. \tag{5.32}$$

Since F depends on the gradient of T, we might as a rough approximation set F_1 proportional to dT_1/dz. This would give an equation of the form $d^2T_1/dz^2 = i\alpha T_1$, familiar in the theory of conduction of heat—as, for example, in the propagation downwards into the Earth of the annual temperature variation at the Earth's surface. The

solution gives a temperature wave propagated with change of phase but with a very rapid decay of amplitude, the decrement being $e^{-\frac{1}{2}\pi}(=1/4{\cdot}8)$ for a phase-retardation of 90°. It is therefore important—

(1) To show definitely, by comparison of theory and observation, that there is no great difference of amplitude of F_1' and $F_1{}^*$.

(2) To show that there is a material difference between the "radiative blanket" and the "conductive blanket", which makes change of phase without decay of amplitude possible in the former.

The value of F_1', obtained by inserting the close approximation $T_1/\theta_a = P_1 = -(4+\sigma)\xi_1$ in the rigorous formula for the adiabatic region*, is

$$F_1' = \left\{4 - \frac{1}{\theta_a}\frac{8+\sigma}{4+\sigma} + \lambda\right\} T_1', \tag{5.41}$$

where $\sigma = n^2\xi_0/g_0$, and has nearly the same value 1·5 in all Cepheids on account of the approximate constancy of $\Pi\sqrt{\rho}$; λ is the index in the opacity law $k \propto P/T^\lambda$, and may be taken to be 4·5. Setting $\theta_a = 0{\cdot}4$, corresponding to $\Gamma_a = \frac{5}{3}$, (5.41) gives

$$F_1' = 4{\cdot}18T_1' = -9{\cdot}2\xi_1'. \tag{5.42}$$

For ξ_1' we may substitute the observed surface value $\xi_1{}^*$; this has a fairly constant mean value 0·055 independent of the period. Then $F_1' = 0{\cdot}506$, corresponding to a range of about $1^{m}{\cdot}1$ bolometric. The mean observed range is about $0^{m}{\cdot}8$. Thus the loss of amplitude between F_1' and $F_1{}^*$, though appreciable, is not very large.

We next examine the equations in the non-adiabatic layer to see why the loss of amplitude is so small. From the equation of radiative equilibrium

$$H = \frac{-ac}{3k\rho}\frac{dT^4}{d\xi} = -\frac{ac\xi^2}{3k\rho_0\xi_0{}^2}\frac{dT^4}{d\xi_0}$$

we obtain by logarithmic differentiation

$$H_1 = 2\xi_1 - k_1 + 4\frac{d(T_0{}^4T_1)}{dT_0{}^4} \tag{5.51}$$

$$= 2\xi_1 - (P_1 - 4{\cdot}5T_1) + 4T_1 + dT_1/d\,(\log T_0).$$

Then, using $P_1 = -(4+\sigma)\xi_1 = -5{\cdot}5\xi_1$,

$$F_1 = H_1 + 2\xi_1 = 9{\cdot}5\xi_1 + 8{\cdot}5T_1 + dT_1/d\,(\log T_0). \tag{5.52}$$

This value is to be inserted in (5.32). Although the ξ_1 term in (5.52) is large, it is not difficult to show that its gradient is small, and it contributes little to dF_1/dz. The important feature is the large term $8{\cdot}5T_1$ which has no counterpart in the conduction problem, where F depends on the gradient of T and not at all on T itself. This is the material difference referred to in (2). If we assume that this term provides the main part of dF_1/dz, (5.32) gives

$$\frac{dT_1}{dz} = \frac{in}{8{\cdot}5F_0}T_1,$$

so that

$$T_1 \propto e^{inz/8{\cdot}5F_0}$$

and the phase of T_1, but not the amplitude, changes with z. The omitted term $dT_1/d\,(\log T_0)$ will occasion some change of amplitude, but it is not likely to be large.

There is no definite limit to the amount of phase-retardation given by (5.32) and (5.52), and ϕ may even exceed 360° if the region of low θ is sufficiently extensive. The

* *M.N.*, **87**, 543, equation (5.25).

fact that the observed values of ϕ are near 90° depends on a quite different consideration, namely the condition for minimum dissipation (5.2). In (5.2) θ_n has been assumed uniform over the whole non-adiabatic region. It must therefore be regarded as a compromise quantity depending on the extent as well as the actual θ of the critical layer. After ϕ reaches 180° both factors of $\theta_n(1-\cos\phi)$ diminish; the maximum therefore occurs between $\phi=0°$ and $\phi=180°$. In a general way we should expect the maximum to be fairly near $\phi=90°$; but it would require a much more elaborate investigation to establish this definitely.

For our numerical calculations we shall take the amplitudes of F_1' and F_1^* equal and $\phi=90°$. Then the semi-amplitude of $F_1'-F_1^*$ is $F_1^*\sqrt{2}$; and the difference between maximum and minimum heat-content of the blanket is $F_0F_1^*\sqrt{2}\Pi/\pi$, where Π is the period. The corresponding quantity per sq. cm. of surface is

$$\frac{\sqrt{2}}{\pi}H_0F_1^*\Pi. \tag{5.6}$$

The blanket has to store this as ionisation energy, its heat capacity being otherwise small. It must therefore contain the right amount of hydrogen per sq. cm. of surface to take up the energy (5.6). In § 7 we shall examine whether this condition can be fulfilled.

The low value of θ makes the critical layer convectively unstable, and it may be suggested that the convection currents provide an alternative mode of energy storage, their speed increasing and decreasing in the course of the pulsation. But since the convective instability is a result of the ionisation, it is scarcely likely that the secondary energy changes of the currents can be as great as the primary energy changes of the ionisation. In any case upper limits to the possible speed of the convection currents can be found, which make it clear that their energy storage is unimportant.

6. *The Period-Luminosity Relation.*—We shall try to determine the form of the period-luminosity relation by the principle of homology; that is to say, we assume that the Cepheids form a series of stars whose outer layers are homologous with respect to those characteristics on which (according to the foregoing theory) pulsatory instability depends. We employ as variates the (steady) radiation flux H and surface-gravity g; a relation between these is equivalent to a mass-density relation, and hence to a period-luminosity relation.

By (5.6) the quantity of hydrogen needed in the critical layer is proportional to $H\Pi$, and its weight to $H\Pi g$. We have therefore to secure that in different Cepheids the pressure P_x at points with the same degree of hydrogen ionisation x is proportional to $H\Pi g$. In pulsation the equilibrium value x will be replaced by an oscillation between limits x_1 and x_2. We are assuming that these limits depend only on the amplitude F_1 or T_1 of the pulsation, and not on the temperature and pressure at the level x which will be different in different stars; it is found that this assumption is approximately true.

Taking the approximate mass-luminosity law $L\propto M^3$, and the theoretical period-density law $\Pi\propto\rho^{-\frac{1}{2}}$, we easily find that $H\Pi g\propto H^{\frac{5}{8}}g^{\frac{1}{8}}$, so that the homology condition is

$$P_x\propto H^{\frac{5}{8}}g^{\frac{1}{8}}. \tag{6.1}$$

Assuming radiative equilibrium with k varying as $\rho/T^{\frac{7}{2}}$, we have (at not too small optical depth) $P\propto H^{-\frac{1}{2}}g^{\frac{1}{2}}T^{4\cdot25}$, so that

$$P_x\propto H^{-\frac{1}{2}}g^{\frac{1}{2}}T_x^{4\cdot25}. \tag{6.2}$$

By the ionisation formula P_x varies very rapidly with T_x, *e.g.* as the 15th power at 12,000°, and as the 10th power at 20,000°. We are concerned with temperatures between these limits (§ 7), and may conveniently adopt

$$P_x\propto T_x^{12\cdot75}. \tag{6.3}$$

Eliminating P_x and T_x between the relations (6.1), (6.2) and (6.3), we obtain

$$H \propto g^{\frac{1}{3}}. \tag{6.4}$$

For comparison we give the corresponding result when k is simply a constant, as when the absorption is due entirely to free electron scattering. Then (6.2) is replaced by

$$P_x \propto H^{-1} g T_x^4. \tag{6.2a}$$

Combined with (6.1) and (6.3), this gives

$$H \propto g^{0.52}. \tag{6.4a}$$

From calculations made in an earlier paper * for different points on the observed period-luminosity and period-spectrum curves, the empirical relation between H and g can be obtained. For the range of period from 1·4 to 39 days, I find $H \propto g^{0.45}$, which may be considered to be in reasonably good agreement with either (6.4) or (6.4a).

7. *Energy Storage in the Critical Layer.*—The ionisation energy of a hydrogen atom is equal to the mean translatory energy of a particle at 105,000°. In typical conditions two-thirds of the ionisation occurs in a temperature range of 20 per cent., say from 10,000° to 12,000°. Thus the specific heat of hydrogen at the critical stage of ionisation may well be 30 or 40 times its ordinary value.

The ionisation formula for hydrogen, with numerical values inserted, is conveniently written

$$\log \frac{P}{1+f} = 11{\cdot}602 - \frac{68000}{T} - \frac{5}{2} \log \frac{68000}{T} + \log \frac{1-x}{x}, \tag{7.1}$$

where f is the proportion of ions (of all elements) to electrons, so that $P/(1+f)$ is the electron pressure P_e. The logarithms are to the base 10.

We define (conventionally) the inner and outer boundaries of the critical layer to be the depths at which $x = \frac{5}{6}$, $\frac{1}{6}$ in the undisturbed star, and denote the corresponding pressures and temperatures by P_i, P_o, T_i, T_o. The mass of material per sq. cm. in the critical layer is then $(P_i - P_o)/g$. On any reasonable estimate of the law of connection of P and T in this part of the star, P_i is several times greater than P_o; and the mass may with sufficient approximation be taken to be P_i/g. If X is the abundance of hydrogen, the mass of hydrogen is XP_i/g.

We consider an oscillation of amplitude $T_1 = 0{\cdot}1$, so that $F_1 = H_1 = 0{\cdot}4$.† The 20 per cent. range of temperature ionises and de-ionises $\frac{2}{3}$ of the hydrogen at the middle of the critical layer. We therefore equate the ionisation energy of $\frac{2}{3}XP_i/g$ grams of hydrogen to the energy storage given by (5.6).

TABLE I

	A	*B*
Mass	5	18 × ⊙
Period	1·42	38·8 days
Mean Eff. Temp.	6500	4030°
g	1280	18·6 cm. sec.$^{-2}$
$H_0 H_1$	$4{\cdot}05 \cdot 10^{10}$	$6{\cdot}04 \cdot 10^{9}$ erg cm.$^{-2}$ sec.$^{-1}$
Energy stored	$2{\cdot}23 \cdot 10^{15}$	$9{\cdot}13 \cdot 10^{15}$ erg cm.$^{-2}$
Mass of hydrogen	257	1050 gm. cm.$^{-2}$
Weight of hydrogen	$3{\cdot}30 \cdot 10^{5}$	$1{\cdot}95 \cdot 10^{4}$ dyne cm.$^{-2}$
P_i	$6{\cdot}6 \cdot 10^{5}$	$3{\cdot}9 \cdot 10^{4}$ dyne cm.$^{-2}$
T_i	17,700	14,050°
$\bar{k}_i$	0·142	0·093

* *M.N.*, **92**, 480, Table III (1932).

† The approximation $F_1 = H_1 = 4T_1$ is rough, but is adequate for our purpose. We can justify $F_1 = 4T_1$ at the inner boundary of the layer by (5.42). Strictly $F_1 = H_1 + 2\xi_1$; but, since H_1 and ξ_1 differ 90° in phase, the amplitudes of F_1 and H_1 are not much different.

The above results (Table I) are found for a short-period and long-period Cepheid for which the principal data were computed in an earlier paper.*

In the following explanation the numbers refer to star A. The energy stored, $2{\cdot}23 \cdot 10^{15}$ ergs per sq. cm. of surface, is calculated by (5.6). This is equal to $\frac{2}{3}$ of the ionisation energy of 257 gm. of hydrogen. Assuming a hydrogen abundance of 50 per cent., the corresponding mass of material is 514 gm., and the pressure P_i is taken to be the weight of this (P_o being relatively small). The electron pressure has been taken to be $\frac{1}{2}P_i$, corresponding to one ion per electron, since $\frac{5}{6}$ of the hydrogen is ionised at the level i. Then T_i is determined from (7.1) with $x=\frac{5}{6}$. The meaning of $\bar{k}_i$ will be explained in § 8.

If the existing equilibrium theory of the outer layers were adequate to determine the pressure-temperature relation at this depth in the star, we should know independently the pressure at the depth corresponding to the temperature 17,700°, and could test whether the above calculated value $6{\cdot}6 \cdot 10^5$ agreed. In particular, if the equilibrium theory showed conclusively that the pressure at that temperature was substantially less than $6{\cdot}6 \cdot 10^5$, it would mean that there is not enough hydrogen at a critical stage of ionisation to perform the task allotted to it in our theory.

The most suitable material for comparison is Unsöld's computation for a model atmosphere on the Sun.† This gives a pressure $5{\cdot}10^3$ in the Sun's photosphere; for the star A it would be rather less. To agree with our result P must vary as T^6, or thereabouts, between the photosphere and the level i. At first sight this variation does not seem extravagant; for example, in the standard model of the deep interior $P \propto T^4$. But there are circumstances which make the radiative gradient $d(\log P)/d(\log T)$ abnormally low in the region below the photosphere; and, in fact, Unsöld's table gives $P=4{\cdot}10^4$ at our temperature T_i. Since the result for star A would be lower, this is a wide discrepancy.

The assumption of adiabatic equilibrium would be more favourable, since it gives a steeper ascent of P. It appears that between 7800° and 14,000° the adiabatic gradient $d(\log P)/d(\log T)$ is greater than 5 and has a maximum of nearly 10.‡ This would suit very well our values of P_i and T_i. But adiabatic equilibrium implies that most of the outflowing heat is transported across the critical layer by convection, and involves velocities of the currents which are quite out of the question.

The large discrepancy has led me to re-examine the conditions in a convective layer. The results of the discussion are given in an accompanying paper, and will be applied in the next section to the problem here raised.

8. *Convection in the Critical Layer.*—The difficulty encountered in § 7 can be put most forcibly in the following way. Assuming radiative equilibrium, the optical depth τ_i of the level i is given by

$$T_i^4 = \tfrac{1}{2} T_e^4 (1 + \tfrac{3}{2}\tau_i).$$

With $T_i = 17{,}700°$, $T_e = 6500°$, we obtain $\tau_i = 72{\cdot}7$. Since the mass of material to this depth is 514 gm. cm.$^{-2}$, the mean opacity $\bar{k}_i$ is $72{\cdot}7/514 = 0{\cdot}142$. For the star B we find similarly $\bar{k}_i = 0{\cdot}093$.

These values of $\bar{k}_i$ are scarcely more than would result from free-electron scattering alone. The scattering coefficient for material containing one free electron for every two units of atomic weight is 0·2; so that 0·1 would correspond to a column which is half hydrogen on the average half ionised. To obtain our value of P_i it would be necessary to assume the absence of all other contributions to the opacity.

In the accompanying paper § I have found that the convection leads to a complete break-down of ionisation equilibrium, and have traced some of the consequences of this.

* *M.N.*, **92**, 480, 1932. These are not actual stars, but mean objects which precisely obey Shapley's period-luminosity and period-spectrum curves.

† *Physik der Sternatmosphären*, p. 144, Table XXXVI.

‡ *Ibid.*, p. 382, fig. 125.

§ Quoted as "I", *M.N.*, **101**, 177, 1941.

One result is a substantial reduction of opacity in the convective region (I, § 4). The reduction is insufficient; but it helps to close the gap between the Cepheid calculation and Unsöld's calculation of P_i. There remains a somewhat reduced discrepancy which, I think, is accounted for as follows:—

In Table I we found the value of T_i corresponding to P_i by the equilibrium ionisation formula (7.1). The result of convection is that this formula no longer applies. The neutral hydrogen convected by the downward current reaches the level 17,700° too quickly to be ionised—the proportion ionised being perhaps no more than $\frac{1}{15}$ (I, § 4). The limit $x_i = \frac{5}{6}$ is therefore depressed to a much higher temperature level. There seems to be no great difficulty in supposing it to fall near 30,000°.

I think therefore that, partly by the reduced opacity and partly by the deeper extension of the critical layer, an adequate amount of hydrogen is obtained. At any rate the close consideration of the effects of convection reveals such considerable modifications that we need not condemn the present theory of the cause of Cepheid pulsation on account of disagreement with the crude equilibrium calculations.

In § 5 we assumed θ to be very small in the blanketing layer; if it is not small the efficiency of the blanket is much reduced. But we have now introduced a flattening of the ionisation gradient by convection, which would increase θ perhaps to the value $\frac{1}{4}$, corresponding to $\Gamma = \frac{4}{3}$. It may be objected that we cannot have it both ways. But, as a matter of fact, we can have it both ways. Γ, being a function of the speed of change, is not the same constant in the two investigations; and θ may well be $\frac{1}{4}$ for the purpose of convection and $\frac{1}{12}$ for the purpose of pulsation. The pulsation is a relatively slow variation which allows time for the ionisation to adjust itself to the altered temperature and pressure. The combination of convection with pulsation is a somewhat complicated problem; but in principle each stage of the pulsation can be treated as quasi-static, the speed and extent of the convection currents and the consequent distribution of pressure, temperature and non-equilibrium ionisation being calculated for successive stages independently.

9. *Limit to the Amplitude.*—The amplitude $T_1 = 0{\cdot}1$ was chosen in § 7 because a temperature range of 20 per cent. covers most of the hydrogen ionisation; for example, two-thirds of the ionisation occurs between 10,900° and 13,400° at pressure 10^4, and between 12,600° and 15,900° at pressure 10^5. If we take a smaller amplitude, the required mass of hydrogen is not much altered; there is less radiation to be stored, and the range of x is smaller in much the same proportion. But if we take a larger amplitude, there is more radiation to be stored, and the range of x which is already near its limit is not proportionately increased. Thus a layer, which is just thick enough to blanket oscillations of any amplitude up to $T_1 = 0{\cdot}1$, is not thick enough to blanket stronger oscillations; and beyond $T_1 = 0{\cdot}1$ the dissipation increases rapidly. Accordingly the pulsation, due to the exceptionally low dissipation in a star which has reached the Cepheid stage, will not increase much beyond this limit.

There is thus a natural limit to the pulsation at about $T_1 = 0{\cdot}1$, or $H_1 = 0{\cdot}4$. This gives a range of $0^m{\cdot}8$ bolometric.

It has been pointed out that another natural limit is imposed by the condition that the pressure must not become negative in the course of the pulsation (*I.C.S.*, § 131); but the actual pulsations fall short of this limit. It is forestalled by the new limit, which agrees reasonably well with observation.

10. *Comparison with other Investigations.*—The retardation of phase has recently been treated by M. Schwarzschild.* I think his work is not in conflict with the present conclusions. He shows that the change of phase cannot occur without a modification of hitherto accepted boundary conditions, but does not suggest any particular cause for the modification. Our investigation indicates the low value of θ in the critical layer as the cause. To make a nearer comparison we must distinguish between (*a*) the change of

* *Zeits. f. Astrophysik*, 15, 14, 1938.

phase which occurs in the modified region, and (*b*) the change of phase imposed on the interior solution by the modification of its boundary conditions. Schwarzschild gives a specimen solution which takes it for granted that (*b*) is the main change. I think the dissipation formula (4.3) is decisive against this; unless a considerable part of the change of phase is actually within the region where θ is small, we obtain negative dissipation for the whole star. Doubtless the critical layer disturbs the conditions for a short distance interior to it; but mathematical analysis indicates that such a disturbance dies out rapidly as we go inwards, and it could scarcely reach the adiabatic region. I conclude that the effect (*b*) is negligible compared with (*a*).

All former calculations of the dissipation have been restricted to D_a, and it might seem that these require revision in view of the present results relating to D_n. But the purpose of the calculations has usually been to examine the stability of ordinary stars rather than the instability of Cepheids. The large negative D_n is a transient condition, of which the phase-retardation is a symptom; as soon as we get away from the Cepheids, $-D_n$ is small. Thus the former calculations are valid for the purpose for which they are intended. In regard to the calculations of D_a, I agree with Cowling * that my own and other early calculations underestimated the dissipation, owing to the integrations being carried not far enough towards the outside of the star, where most of the dissipation (positive and negative) occurs.

Summary

A study of the dissipation of the energy of pulsation leads to the conclusion that the quarter-period retardation of the outflow of heat is necessarily associated with an abnormally low adiabatic constant in the non-adiabatic part of the star. This points unmistakably to the critical layer, where hydrogen is in the mid-stage of ionisation, as responsible for the phase-retardation. It is verified that phase change can occur in this region without the rapid decay of amplitude which might have been expected from analogy with conduction problems. For stars in general no fixed amount of phase change is indicated; but in Cepheids it will correspond to an optimum extension of the critical layer which makes the dissipation a minimum, the sharply reduced dissipation being the cause of the pulsatory instability. A certain amount of confirmatory evidence is found. The theory requires an amount of hydrogen in the critical layer which at first sight seems impossibly large; but it is believed that the discrepancy is removed by the study of the effects of convection contained in an accompanying paper.

* *M.N.*, 94, 768, 1934.

THE ELLIPSOIDAL DISTRIBUTION OF STELLAR VELOCITIES

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1. *Introduction.*—In an attempt to account for the phenomenon of star-streaming,* Schwarzschild suggested that the distribution of stellar velocities should be ellipsoidal, instead of the isotropic distribution which is associated with the kinetic theory of gases. The suggested theory was found to account for the observed distribution of proper-motions—the "drift-curves"—quite well, and, on the basis of it, new methods of analysing stellar motions were soon devised, and are still in use.

The amount of material available for such analyses is now much greater, but good agreement between theory and observation is still found. Yet it must be stressed that the ellipsoidal theory does not fit the observations so remarkably closely that all other types of velocity-distribution must be ruled out. Further, though the theoretical distribution may represent the observational material adequately, unless it is exact it may lead to false conclusions when combined with the dynamical laws of motion.

However, as the agreement was so satisfactory, it became a matter of some importance to discover whether such a distribution was dynamically possible. Papers on this subject have appeared beneath the names of Eddington†, Jeans‡, Oort§, Shiveshwarkar||, Chandrasekhar¶ and others. The general conclusions may briefly be summed up as follows: There would appear to be a completely general solution of the problem, a solution which fits the observed features of three unequal axes for the velocity-ellipsoid and a deviation of the vertex of star-streaming from the direction of the galactic centre. The complete requirements of this solution, such as the form of the gravitational potential, have not previously been determined. Some particular solutions have been found, but these systems do not possess all the properties of the observed ellipsoids.

The present paper, which follows up these conclusions, is divided into three parts—all concerned only with systems in a steady state possessing a generalized ellipsoidal distribution of velocities. In the first place it is shown that, with two simple restrictions, the system must have complete rotational symmetry. The two restrictions are that the mean motion of the system is not zero at every point, and that the system is not spherically symmetrical. In general, the velocity-ellipsoid will have three unequal axes, but the vertex of star-streaming can only deviate from the direction to the galactic centre in latitude and not in longitude.

The second part is an investigation of the form of the gravitational potential necessary to maintain such a system. The general case is given, in addition to the two particular cases, only one of which had been solved before. This completes for the first time the problem of the steady ellipsoidal distribution.

Finally it is shown that no such system of stars of equal mass can exist unless there is some external field of force to supplement that due to the mutual attractions of the separate bodies. There is the possibility that a system in which different types of stars have slight differences in velocity-distribution may be able to exist entirely under its own gravitational potential. However, it would seem rather improbable that a system which could not exist if its component masses were all equal, can yet be in equilibrium because the separate masses are unequal.

2. *The Gravitational Potential has Rotational Symmetry.*—In order to describe the

* *Göttingen Nachrichten*, p. 614, 1907. † *M.N.*, 76, 37, 1915. ‡ *M.N.*, 76, 70, 1915.
§ *B.A.N.*, 4, No. 159, 1928. || *M.N.*, 96, 749, 1936. ¶ *Ap. J.*, 90, 1, 1939.

system of stars under consideration, we adopt any convenient non-accelerated Cartesian frame of space-co-ordinates, x, y, z, and corresponding velocity-components, u, v, w. Then the velocity-distribution function, $f(x, y, z, u, v, w)$, is such that the number of stars in a volume of space, $dxdydz$, surrounding the point (x, y, z), and of which the velocity-components lie between u and $u+du$, v and $v+dv$, w and $w+dw$, is the quantity,

$$f du dv dw dx dy dz.$$

The velocity-distribution function is not arbitrary, but must satisfy an equation of continuity; this is a generalized Boltzmann equation, as found in the kinetic theory of gases, and leads to the equation :

$$u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}+\frac{du}{dt}\frac{\partial f}{\partial u}+\frac{dv}{dt}\frac{\partial f}{\partial v}+\frac{dw}{dt}\frac{\partial f}{\partial w}=0. \tag{1}$$

The three components of acceleration, $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$, are those produced by the mutual gravitational attractions of the stars. Jeans and others * have shown that the effects of the attraction of any individual star on any other are quite negligible in comparison with the gravitational field of the entire system. Hence the components of acceleration in (1) will be replaced by the appropriate derivatives of a potential function, Ω—the gravitational potential due to the whole system. Thus we write

$$u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}+\frac{\partial \Omega}{\partial x}\frac{\partial f}{\partial u}+\frac{\partial \Omega}{\partial y}\frac{\partial f}{\partial v}+\frac{\partial \Omega}{\partial z}\frac{\partial f}{\partial w}=0. \tag{2}$$

In general, the distribution-function will involve the time explicitly, and the left-hand side of (2) is not equal to zero but to $\left(-\frac{\partial f}{\partial t}\right)$; however, it will be assumed in this investigation that the system is in a steady state, so that f and Ω are independent of the time.

The form in which the function f contains the velocity-components is assumed and is written

$$f \equiv f\{\phi+pu+qv+rw+au^2+bv^2+cw^2+2fvw+2gwu+2huv\}. \tag{3}$$

In this the quantities ϕ, p, q, r, a, b, c, f, g, h are all independent of the velocity-components, but will, in general, contain the space co-ordinates. It should be noted that the symbol f is used in two senses, both as a coefficient and also as the distribution-function itself. However, the two senses are hardly likely to be confused. About the function f it is only supposed that it is a function of a single argument, that it is non-negative, and that it tends to zero as its argument takes large positive values. The argument itself is a positive definite form in u, v, w for all values of the space co-ordinates, otherwise there would be a finite number of stars with an infinite velocity. This type of velocity-distribution is a generalization of the original ellipsoidal type, in which the function f is simply the exponential function, the argument being the negative index. The problem is no more complicated in this slightly more general case.

The expression for f, (3), is substituted in (2), and a polynomial of the third degree in u, v, w is obtained. Now (2) must be satisfied for all values of u, v, w, x, y, z, so the coefficients of the different velocity-terms may be selected and equated to zero. In particular, we have from the six second-order terms in u, v, w the six equations,

$$\frac{\partial p}{\partial x}\equiv\frac{\partial q}{\partial y}\equiv\frac{\partial r}{\partial z}\equiv 0, \qquad \frac{\partial r}{\partial y}+\frac{\partial q}{\partial z}\equiv\frac{\partial p}{\partial z}+\frac{\partial r}{\partial x}\equiv\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\equiv 0.$$

* Smart, *Stellar Dynamics*, p. 319, 1938.

The solution of these equations is

$$p = p_0 + \eta z - \zeta y, \qquad q = q_0 + \zeta x - \xi z, \qquad r = r_0 + \xi y - \eta x.$$

Here p_0, q_0, r_0, ξ, η, ζ are arbitrary constants. Now the directions of the axes of co-ordinates were chosen at random, as were also the co-ordinate planes; it will now be more convenient to choose a new frame of axes so inclined to the old that the vector (ξ, η, ζ) in the old frame becomes the vector $(0, 0, \omega)$ in the new. The quantity, ω, is given by the relation

$$\omega^2 = \xi^2 + \eta^2 + \zeta^2.$$

In this new frame the coefficients p, q, r simply become

$$p = p_0' - \omega y, \qquad q = q_0' + \omega x, \qquad r = r_0'.$$

The quantities p_0', q_0', r_0' are constants, being linear combinations of the old p_0, q_0, r_0. They are not necessarily zero, but p_0' and q_0' can always be made to vanish by a convenient choice of the planes, $x = 0$, $y = 0$. Hence, unless ξ, η, ζ are identically zero, it can always be arranged to have

$$p = -\omega y, \qquad q = +\omega x, \qquad r = r_0.$$

Now there is one further relation between p, q, r—one which is obtained by substituting (3) in (2) and selecting the terms which are completely independent of the velocity-components. They give

$$p\frac{\partial \Omega}{\partial x} + q\frac{\partial \Omega}{\partial y} + r\frac{\partial \Omega}{\partial z} = 0. \tag{4}$$

In this the values of p, q, r are substituted, and we have

$$\omega\left(x\frac{\partial \Omega}{\partial y} - y\frac{\partial \Omega}{\partial x}\right) + r_0\frac{\partial \Omega}{\partial z} = 0. \tag{5}$$

The general solution of this has been given by Chandrasekhar *, and leads to a type of system which he describes as possessing helical symmetry. Unfortunately such systems are infinite in extent and so are outside the scope of this work. That such systems are infinite is seen most easily from equation (5); along the axis of z we must have $\frac{\partial \Omega}{\partial z} = 0$, unless, of course, r_0 is identically zero. But for a finite system of mass M we have, at very distant points on the z-axis, $\frac{\partial \Omega}{\partial z} \sim -M/z^2$. Hence, if $\frac{\partial \Omega}{\partial z}$ is to vanish at all points on the axis, the system cannot be finite. It is therefore necessary to take the particular case, $r_0 = 0$. Equation (5) now becomes

$$x\frac{\partial \Omega}{\partial y} - y\frac{\partial \Omega}{\partial x} = 0. \tag{6}$$

This necessarily follows, for it is impossible for r_0 and ω to vanish simultaneously, since that would make p, q, r all zero, and would thus be contrary to the assumption that the mean motion of the system is not everywhere zero.

There is one exception to the general conclusion just reached †; this is the case, already mentioned, in which ξ, η, ζ are identically zero. In this case (4) becomes

$$p_0\frac{\partial \Omega}{\partial x} + q_0\frac{\partial \Omega}{\partial y} + r_0\frac{\partial \Omega}{\partial z} = 0,$$

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* *Ap. J.*, **90**, 53, 1939.

† Lindblad has established that the gravitational potential has rotational symmetry, but does not point out this apparent exception, in *Stockholms Obs. Ann.*, **13**, No. 5, 8, 1940.

or, by a suitable change in the directions of the axes, $\frac{\partial\Omega}{\partial z}=0$. This again necessarily follows, for it is not permissible for the six quantities p_0, q_0, r_0, ξ, η, ζ all to vanish simultaneously. But it is clearly impossible for $\frac{\partial\Omega}{\partial z}$ to vanish at all points of a finite system, so that the only systems which are finite and in which the mean motion is not everywhere zero are those for which equation (6) is satisfied.

It is seen to be advantageous to transfer to cylindrical polar co-ordinates, with the axis of the cylinder along the axis of z. The new space co-ordinates are R, θ, z and there are corresponding components of linear velocity, Π, Θ, Z. The terms in (3) which are linear in the velocity-components reduce simply to $\omega R\Theta$, and in the new co-ordinates (6) becomes

$$\frac{\partial\Omega}{\partial\theta}=0. \tag{7}$$

This has established that for systems which are finite, and which have a non-zero mean motion, the gravitational potential possesses rotational symmetry.

3. *Such Systems possess Complete Rotational Symmetry.*—In the cylindrical co-ordinates of the previous section the equations of motion become

$$\frac{d\Pi}{dt}=\frac{\partial\Omega}{\partial R}+\frac{\Theta^2}{R},\qquad \frac{d\Theta}{dt}=-\frac{\Pi\Theta}{R},\qquad \frac{dZ}{dt}=\frac{\partial\Omega}{\partial z}.$$

The equation of continuity now takes the form,

$$\Pi\frac{\partial f}{\partial R}+\frac{\Theta}{R}\frac{\partial f}{\partial\theta}+Z\frac{\partial f}{\partial z}+\left(\frac{\partial\Omega}{\partial R}+\frac{\Theta^2}{R}\right)\frac{\partial f}{\partial\Pi}-\frac{\Pi\Theta}{R}\frac{\partial f}{\partial\Theta}+\frac{\partial\Omega}{\partial z}\frac{\partial f}{\partial Z}=0. \tag{8}$$

Similarly, the assumed velocity-distribution is written

$$f\equiv f\{\phi+\omega R\Theta+a\Pi^2+b\Theta^2+cZ^2+2f\Theta Z+2gZ\Pi+2h\Pi\Theta\}. \tag{9}$$

When (9) is substituted in (8) the resulting polynomial is of the third order in the velocity-components. By selecting the ten coefficients of the third-order terms we can determine the six coefficients of the velocity-ellipsoid, in terms of the space co-ordinates, R, θ, z, and twenty arbitrary constants. The full details of the solution need not be given here, as they have already been published by Chandrasekhar.* The expressions for the six coefficients are

$$\begin{aligned}
a&=\alpha_0-2z\alpha_1+z^2\alpha_3+(\beta_0-2z\beta_1+z^2\beta_3)\sin 2\theta\\
&\qquad+(\gamma_0-2z\gamma_1+z^2\gamma_3)\cos 2\theta,\\
b&=\alpha_0-2z\alpha_1+z^2\alpha_3-(\beta_0-2z\beta_1+z^2\beta_3)\sin 2\theta-(\gamma_0-2z\gamma_1+z^2\gamma_3)\cos 2\theta\\
&\qquad+R^2\alpha_2-2R(\beta_5+z\beta_6)\sin\theta-2R(\gamma_5+z\gamma_6)\cos\theta,\\
c&=\alpha_4-2R(\beta_2\sin\theta+\gamma_2\cos\theta)+R^2(\alpha_3+\beta_3\sin 2\theta+\gamma_3\cos 2\theta),\\
f&=R\alpha_5+(\beta_4+z\beta_2-R^2\beta_6)\cos\theta-(\gamma_4+z\gamma_2-R^2\gamma_6)\sin\theta\\
&\qquad+R(\beta_1-z\beta_3)\cos 2\theta-R(\gamma_1-z\gamma_3)\sin 2\theta,\\
g&=R(\alpha_1-z\alpha_3)+R(\beta_1-z\beta_3)\sin 2\theta+R(\gamma_1-z\gamma_3)\cos 2\theta\\
&\qquad+(\beta_4+z\beta_2)\sin\theta+(\gamma_4+z\gamma_2)\cos\theta,\\
h&=R(\beta_5+z\beta_6)\cos\theta-R(\gamma_5+z\gamma_6)\sin\theta+(\beta_0-2z\beta_1+z^2\beta_3)\cos 2\theta\\
&\qquad-(\gamma_0-2z\gamma_1+z^2\gamma_3)\sin 2\theta.
\end{aligned}$$

In these expressions the quantities α, β, γ, twenty of them in all, are constants of integration. It can now be shown that the fourteen β's and γ's must all be zero, so

* *Ap. J.*, 90, 119, 1939.

that the coefficients are independent of θ. This follows from a consideration of three further equations, obtained by selecting the coefficients of the three terms linear in the velocity-components when (9) is substituted in (8). The equations are

$$\left.\begin{aligned} \frac{\partial\phi}{\partial R}+2a\frac{\partial\Omega}{\partial R}+2g\frac{\partial\Omega}{\partial z}&=0,\\ \frac{1}{R}\frac{\partial\phi}{\partial\theta}+2h\frac{\partial\Omega}{\partial R}+2f\frac{\partial\Omega}{\partial z}&=0,\\ \frac{\partial\phi}{\partial z}+2g\frac{\partial\Omega}{\partial R}+2c\frac{\partial\Omega}{\partial z}&=0. \end{aligned}\right\} \tag{10}$$

In these equations the quantity ϕ can be eliminated by the cross-differentiation of any pair, and so three equations, linear in the first and second derivatives of Ω with respect to R and z, are obtained. In accordance with (7) there are no derivatives with respect to θ. Thus Ω satisfies the equations,

$$\frac{\partial a}{\partial\theta}\frac{\partial\Omega}{\partial R}+\frac{\partial g}{\partial\theta}\frac{\partial\Omega}{\partial z}=\frac{\partial}{\partial R}\left(Rh\frac{\partial\Omega}{\partial R}\right)+\frac{\partial}{\partial R}\left(Rf\frac{\partial\Omega}{\partial z}\right), \tag{11}$$

$$\frac{\partial g}{\partial\theta}\frac{\partial\Omega}{\partial R}+\frac{\partial c}{\partial\theta}\frac{\partial\Omega}{\partial z}=R\frac{\partial}{\partial z}\left(h\frac{\partial\Omega}{\partial R}\right)+R\frac{\partial}{\partial z}\left(f\frac{\partial\Omega}{\partial z}\right), \tag{12}$$

$$\frac{\partial}{\partial z}\left(a\frac{\partial\Omega}{\partial R}\right)+\frac{\partial}{\partial z}\left(g\frac{\partial\Omega}{\partial z}\right)=\frac{\partial}{\partial R}\left(g\frac{\partial\Omega}{\partial R}\right)+\frac{\partial}{\partial R}\left(c\frac{\partial\Omega}{\partial z}\right). \tag{13}$$

Since a, c, f, g, h only contain θ in one of the forms, $\cos 2\theta$, $\sin 2\theta$, $\cos\theta$, $\sin\theta$, and since each of these only occurs linearly, it is possible to make five equations from each of (11), (12), (13), by selecting the coefficients of each of those four forms, and also the terms which are independent of θ. The resulting fifteen equations for Ω will clearly impose some restrictions on the twenty constants of integration, α, β, γ.

The restriction turns out to be that the β's and γ's must be identically zero. To give a full proof of this would take up much space, as there are fourteen constants and almost every one of them requires separate treatment. The process will therefore be illustrated by showing that the three constants, β_0, β_1, β_3, are identically zero.

In equations (11) and (12) the coefficients of $\cos 2\theta$ are selected, and they give the two equations,

$$R(\beta_0-2z\beta_1+z^2\beta_3)\frac{\partial^2\Omega}{\partial R^2}-(\beta_0-2z\beta_1+z^2\beta_3)\frac{\partial\Omega}{\partial R}+R^2(\beta_1-z\beta_3)\frac{\partial^2\Omega}{\partial R\partial z}=0, \tag{14}$$

$$R(\beta_0-2z\beta_1+z^2\beta_3)\frac{\partial^2\Omega}{\partial R\partial z}-4R(\beta_1-z\beta_3)\frac{\partial\Omega}{\partial R}+R^2(\beta_1-z\beta_3)\frac{\partial^2\Omega}{\partial z^2}-3R^2\beta_3\frac{\partial\Omega}{\partial z}=0. \tag{15}$$

Of these, the first is at once integrable, to give

$$(\beta_0-2z\beta_1+z^2\beta_3)\frac{1}{R}\frac{\partial\Omega}{\partial R}+(\beta_1-z\beta_3)\frac{\partial\Omega}{\partial z}=(\beta_1-z\beta_3)F_1(z). \tag{16}$$

Equation (15) is divided by R^2 and then subtracted from the derivative of (16) with respect to z; it follows that

$$\frac{2}{R}(\beta_1-z\beta_3)\frac{\partial\Omega}{\partial R}+2\beta_3\frac{\partial\Omega}{\partial z}=(\beta_1-z\beta_3)F_1'(z)-\beta_3F_1(z). \tag{17}$$

Here, and throughout, the prime symbol is used to denote the derivative of a function of a single variable with respect to that variable. Now (16) is multiplied by $2\beta_3$, and (17) is multiplied by $(\beta_1-z\beta_3)$ and subtracted, so that

$$2(\beta_0\beta_3-\beta_1{}^2)\frac{1}{R}\frac{\partial\Omega}{\partial R}=3\beta_3(\beta_1-z\beta_3)F_1(z)-(\beta_1-z\beta_3)^2F_1'(z).$$

Then, unless $\beta_0\beta_3=\beta_1{}^2$, this can be integrated to give

$$\Omega=R^2F_2(z)+F_3(z). \tag{18}$$

This is seen to be an impossible solution for the gravitational potential of a finite system whether the function $F_2(z)$ is identically zero or not. It is therefore necessary to take the only alternative, $\beta_0\beta_3=\beta_1{}^2$.

In this case (16) is multiplied by β_3 and divided by $(\beta_1-z\beta_3)$; this gives

$$(\beta_1-z\beta_3)\frac{1}{R}\frac{\partial\Omega}{\partial R}+\beta_3\frac{\partial\Omega}{\partial z}=\beta_3F_1(z). \tag{19}$$

From (17) and (19) it follows that $F_1(z)$ satisfies the equation,

$$(\beta_1-z\beta_3)F_1'(z)-3\beta_3F_1(z)=0.$$

When integrated this gives

$$F_1(z)=K/(\beta_1-z\beta_3)^3.$$

We now return to (16) and change the independent variables from (R, z) to (r, χ), where the new co-ordinates are defined by

$$r^2=R^2+(z-\beta_1/\beta_3)^2,$$
$$\tan\chi=(z-\beta_1/\beta_3)/R.$$

In the new co-ordinates (16) becomes

$$\frac{\partial\Omega}{\partial\chi}=-K\cos\chi/\beta_3{}^2r^2\sin^3\chi.$$

This can be immediately integrated to give

$$\Omega=F_2(r)+K/2(\beta_1-z\beta_3)^2.$$

Then K must be zero, otherwise the gravitational potential is infinite at all points in the plane, $z=\beta_1/\beta_3$. However, if K is zero, the gravitational potential has complete spherical symmetry. But the outstanding feature of the galactic system, and also of the extra-galactic nebulæ, is the great flattening observed. It has therefore been considered quite unnecessary to include any spherical systems in the analysis.

In order to escape this unacceptable result it is not sufficient that the constants, β_0 and β_1, should be zero, for they do not affect the solution. However, if β_3 is zero, then the whole treatment is invalidated. In that case it follows from (17) that

$$\frac{2}{R}\beta_1\frac{\partial\Omega}{\partial R}=\beta_1F_1'(z).$$

This can be directly integrated to give a solution for Ω of the form (18), always provided that β_1 is not zero. Since (18) does not correspond to a genuine solution of the problem, the alternative $\beta_1=0$ has to be taken.

Equation (16) now becomes

$$\beta_0\frac{1}{R}\frac{\partial\Omega}{\partial R}=F_2(z).$$

This also will integrate to give the unacceptable solution (18), unless it is satisfied identically, in the case when β_0 and F_2 are both zero.

It has now been shown that the three constants, β_0, β_1, β_3, must all be zero, if the gravitational potential is not to be spherically symmetrical, nor yet to have infinite

values at finite points. In precisely the same way—although the actual integrations are not usually so easy—it can be shown that all the remaining β's and γ's must also vanish, so that the coefficients of the velocity-ellipsoid are all independent of θ.

Hence, by differentiating the first and last of equations (10) with respect to θ, we obtain that

$$\frac{\partial^2\phi}{\partial R\partial\theta}=\frac{\partial^2\phi}{\partial z\partial\theta}=0\ ;$$

this shows that $\dfrac{\partial\phi}{\partial\theta}$ depends only on θ. But from the second of equations (10) it is seen that

$$\frac{\partial\phi}{\partial\theta}=-2R^2\alpha_5\frac{\partial\Omega}{\partial z}.$$

Since Ω itself is independent of θ, there must be a constant, K, such that

$$-\frac{\partial\phi}{\partial\theta}=K=2R^2\alpha_5\frac{\partial\Omega}{\partial z}.$$

This integrates, to give

$$\Omega=F(R)+Kz/2R^2\alpha_5.$$

Again, this is a solution which cannot correspond to a physical system, and this absurd result can only be avoided by taking the two constants, α_5 and K, to be identically zero. In this case it is seen that ϕ also is independent of θ, and the whole velocity-distribution depends on the variables R, z only.

In particular it should be noted that the coefficients f and h in the velocity-ellipsoid must be identically zero, so that it is only possible for the vertex of star-streaming to deviate from the direction of the galactic centre in galactic latitude. There can be no deviation in longitude.

4. *The Determination of the Gravitational Potential.*—It has now been established that systems with a steady ellipsoidal velocity-distribution which have rotational symmetry for their gravitational potential must have complete rotational symmetry, except for the very special case of systems with spherical symmetry. In the case of complete rotational symmetry the coefficients of the velocity-ellipsoid become

$$\begin{aligned} a&=\alpha_0-2z\alpha_1+z^2\alpha_3, & g&=R(\alpha_1-z\alpha_3),\\ b&=\alpha_0-2z\alpha_1+z^2\alpha_3+R^2\alpha_2, & f&=h=0.\\ c&=\alpha_4+R^2\alpha_3, \end{aligned}$$

Since the argument of the assumed distribution-function must be a positive definite form, it is seen that α_3 cannot be zero unless α_1 is also zero. Without loss of generality, therefore, the plane $z=0$ can always so be chosen that the constant α_1 is zero. The coefficients are now further reduced, and simply are

$$\begin{aligned} a&=\alpha_0+z^2\alpha_3, & g&=-Rz\alpha_3,\\ b&=\alpha_0+z^2\alpha_3+R^2\alpha_2, & f&=h=0.\\ c&=\alpha_4+R^2\alpha_3, \end{aligned}$$

Of the three equations (10), the second is now satisfied automatically; the first and third give

$$\frac{\partial\phi}{\partial R}+2(\alpha_0+z^2\alpha_3)\frac{\partial\Omega}{\partial R}-2Rz\alpha_3\frac{\partial\Omega}{\partial z}=0, \tag{20}$$

$$\frac{\partial\phi}{\partial z}-2Rz\alpha_3\frac{\partial\Omega}{\partial R}+2(\alpha_4+R^2\alpha_3)\frac{\partial\Omega}{\partial z}=0. \tag{21}$$

As before, the function ϕ can be eliminated by cross-differentiation, and then Ω must satisfy the equation

$$a_3\left\{3z\frac{\partial\Omega}{\partial R}-3R\frac{\partial\Omega}{\partial z}+(z^2-R^2)\frac{\partial^2\Omega}{\partial R\partial z}+Rz\left(\frac{\partial^2\Omega}{\partial R^2}-\frac{\partial^2\Omega}{\partial z^2}\right)\right\}+(a_0-a_4)\frac{\partial^2\Omega}{\partial R\partial z}=0. \qquad (22)$$

This was first given by Oort *, as an equation to be satisfied by the gravitational potential of a system which was *assumed* to possess rotational symmetry. We have just shown that all steady ellipsoidal systems have such symmetry. Oort has given one very special solution of the equation, namely, the one obtained when the constant a_3 happens to be zero, and the constants a_0 and a_4 are equal. In this case it follows from (20) and (21) that

$$\phi=-2a_0\Omega+\text{constant}.$$

The velocity-ellipsoid now becomes a spheroid, with the axes in the directions Π and Z equal. The distribution-function is of the very special type which involves only the integrals of energy and angular momentum; while such distributions are quite possible, they clearly differ from the distribution in the galactic system, in which the velocity-ellipsoid has almost invariably three unequal axes.

It is seen from (22) that if a_3 is zero, then either a_0 and a_4 are equal, or else $\dfrac{\partial^2\Omega}{\partial R\partial z}=0$. Lindblad †, who has paid much attention to distributions containing only the two integrals of energy and angular momentum, obtains systems with three unequal axes by supposing that there is a finite region of space in which $\dfrac{\partial^2\Omega}{\partial R\partial z}$ vanishes. It is impossible for $\dfrac{\partial^2\Omega}{\partial R\partial z}$ to vanish everywhere, for then the solution for Ω is

$$\Omega=F(R)+G(z).$$

This cannot be the potential due to a finite system, for if R is fixed, and z tends to infinity, then Ω should tend to zero; obviously this does not occur, for $F(R)$ remains constant. As Lindblad only supposes that $\dfrac{\partial^2\Omega}{\partial R\partial z}$ vanishes within a finite region, he evades the difficulty, but in place of it there is the question of the continuity of the potential and of the density on crossing from the region where $\dfrac{\partial^2\Omega}{\partial R\partial z}$ vanishes to the region where it does not. The artifice which Lindblad has used seems all the more unnecessary, since it is always possible to obtain three unequal axes by retaining the constant, a_3.

A more general solution of equations (20) and (21) has been found for the case in which a_3 is not zero, but a_0 and a_4 are still equal. To obtain this, it is more convenient to transfer to the co-ordinates (r, χ) of the previous section. They are such that r is the spherical radius, and χ is the latitude. In these variables, equations (20) and (21) become

$$\left\{\frac{R}{r}\frac{\partial}{\partial r}-\frac{z}{r^2}\frac{\partial}{\partial\chi}\right\}(\phi+2a_0\Omega)-2a_3z\frac{\partial\Omega}{\partial\chi}=0,$$

$$\left\{\frac{z}{r}\frac{\partial}{\partial r}+\frac{R}{r^2}\frac{\partial}{\partial\chi}\right\}(\phi+2a_0\Omega)+2a_3R\frac{\partial\Omega}{\partial\chi}=0.$$

When the derivatives of ϕ are separated, it is seen that

$$\frac{\partial}{\partial r}(\phi+2a_0\Omega)=0,$$

$$\frac{\partial}{\partial\chi}(\phi+2a_0\Omega)+2a_3r^2\frac{\partial\Omega}{\partial\chi}=0.$$

* *B.A.N.*, **4**, 276, 1928. † *Stockholms Obs. Ann.*, **13**, No. 5, 10, 1940.

These are directly integrable, giving

$$\phi + 2a_0\Omega = 2a_3 G(\chi),$$
$$\phi + 2a_0\Omega + 2a_3 r^2\Omega = 2a_3 F(r).$$

It therefore follows that

$$\left.\begin{aligned} \Omega &= \{F(r) - G(\chi)\}/r^2, \\ \phi &= 2\{(a_0 + a_3 r^2)G(\chi) - a_0 F(r)\}/r^2. \end{aligned}\right\} \tag{23}$$

It will be seen from this solution that the function F, a function of the spherical radius only, determines the general distribution of the density, while the function G, a function of latitude only, determines the degree of flattening.

The completely general solution of (20) and (21) is of the same nature, but the substitution required to solve them is much more obscure. The constants are such that $a_3 \neq 0$, $a_4 \neq a_0$. It is convenient to define new combinations of them, namely,

$$\gamma = \tfrac{1}{2}(a_0 + a_4), \qquad \beta = \tfrac{1}{2}(a_0 - a_4)/a_3.$$

These constants must not be confused with the β's and γ's of the previous section, which, it has already been shown, must all be zero. Further, new space co-ordinates are adopted, and are defined by the relations

$$R^2 = \tfrac{1}{2}(\xi + \beta)(\eta + \beta)/\beta, \qquad z^2 = -\tfrac{1}{2}(\xi - \beta)(\eta - \beta)/\beta.$$

Then ξ and η are the roots of the quadratic equation,

$$t^2 - t(R^2 + z^2) + \beta(R^2 - z^2 - \beta) = 0.$$

It is clear that one root, say ξ, is always greater than or equal to $|\beta|$, while the absolute value of the other never exceeds $|\beta|$. For large values of r, we have as a good approximation, $\xi = r^2$, $\eta = \beta \cos 2\chi$. Lastly, one of the dependent variables is changed, a new function, Φ, being defined by the relation

$$\Phi = (\tfrac{1}{2}\phi + \gamma\Omega)/a_3.$$

The equations (20) and (21) are now transformed to the new system of co-ordinates, and, when the derivatives are separated, they are found to be connected by the equations,

$$\frac{\partial\Phi}{\partial\eta} + \xi\frac{\partial\Omega}{\partial\eta} = 0, \qquad \frac{\partial\Phi}{\partial\xi} + \eta\frac{\partial\Omega}{\partial\xi} = 0.$$

These can be solved at once to give

$$\Omega = \frac{F(\xi) - G(\eta)}{\xi - \eta},$$
$$\Phi = \frac{\xi G(\eta) - \eta F(\xi)}{\xi - \eta}.$$

The solution is clearly very similar to (23), especially for large values of r, and, as before, F determines the general distribution of density, and G determines the degree of flattening.

If any pair of functions are given for F and G they will determine a distribution of velocities which is not entirely unique—for the actual f of the distribution is still at our disposal—but a distribution in which the variation of the velocity-ellipsoid with the space co-ordinates is unique, as is also the gravitational potential which controls the system.

There is a clear distinction to be drawn between the general solution, just found, and the very special solution which Oort originally gave. The two new solutions given

in this paper each contain two arbitrary functions of a single variable, whereas Oort's solution contained only one arbitrary function, but that was a function of two variables.

5. *The Determination of the Mass-Density.*—The gravitational potential which is required to maintain a system of the type considered has now been found, both in general and particular cases. It is now possible to find what distribution of matter will give rise to such a gravitational potential, for the potential, Ω, and the mass-density, ρ, are connected by Poisson's equation. For a system which has rotational symmetry, Poisson's equation in cylindrical polar co-ordinates becomes

$$\nabla^2\Omega \equiv \frac{\partial^2\Omega}{\partial R^2} + \frac{1}{R}\frac{\partial\Omega}{\partial R} + \frac{\partial^2\Omega}{\partial z^2} = -\rho.$$

This is now transformed to the co-ordinates required in the general solution, namely, ξ and η. The relations between the derivatives are

$$\frac{\partial}{\partial R} \equiv 2R\left\{\frac{\xi-\beta}{\xi-\eta}\frac{\partial}{\partial\xi} + \frac{\eta-\beta}{\eta-\xi}\frac{\partial}{\partial\eta}\right\},$$

$$\frac{\partial}{\partial z} \equiv 2z\left\{\frac{\xi+\beta}{\xi-\eta}\frac{\partial}{\partial\xi} + \frac{\eta+\beta}{\eta-\xi}\frac{\partial}{\partial\eta}\right\}.$$

Finally, when the expression for Ω is substituted, it is found that

$$\nabla^2\Omega = -\rho = 6\frac{\xi F' + \eta G'}{(\xi-\eta)^2} - 2\beta\frac{F'+G'}{(\xi-\eta)^2} + 2\frac{F-G}{(\xi-\eta)^3}(\xi+\eta+2\beta)$$
$$-8\frac{F'(\xi^2-\beta^2)}{(\xi-\eta)^3} + 8\frac{G'(\eta^2-\beta^2)}{(\xi-\eta)^3} + 4\frac{F''(\xi^2-\beta^2)}{(\xi-\eta)^2} + 4\frac{G''(\eta^2-\beta^2)}{(\xi-\eta)^2}. \qquad (24)$$

If the (r, χ) co-ordinates are used, the form of Poisson's equation becomes

$$\nabla^2\Omega = -\rho = \frac{\partial^2\Omega}{\partial r^2} + \frac{2}{r}\frac{\partial\Omega}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2\Omega}{\partial\chi^2} - \frac{\partial\Omega}{\partial\chi}\tan\chi\right).$$

On substituting the expression (23) for Ω we obtain

$$\nabla^2\Omega = -\rho = \{r^2F'' - 2rF' + 2F - 2G - G'\tan\chi + G''\}/r^4. \qquad (25)$$

These two expressions, (24) and (25), give the mass-density, in suitable units, as a function of the co-ordinates.

6. *The Determination of the Particle-Density.*—It has been pointed out that, from its definition, the velocity-distribution function gives the number of stars in a given space-element which have velocities in a given velocity-element; it therefore follows that, by integration with respect to the velocity-components, the total number of particles in a given space-element—that is, the particle-density—is obtained.

In the case considered here two of the three integrations can actually be carried out, and the third can be represented by a suitably defined new function.

It has been shown that the coefficients f and h in the velocity-ellipsoid are identically zero, so that the particle-density, ν, is defined by the equation

$$\nu = \iiint_{\Pi,\,\Theta,\,Z=-\infty}^{+\infty} f\{\phi + \omega R\Theta + a\Pi^2 + b\Theta^2 + cZ^2 + 2g\Pi Z\}d\Pi d\Theta dZ.$$

By the simple substitution, $\Theta^* = \Theta + \omega R/2b$, this is reduced to

$$\nu = \iiint_{\Pi,\,\Theta^*,\,Z=-\infty}^{+\infty} f\{(\phi - \omega^2R^2/4b) + a\Pi^2 + b\Theta^{*2} + cZ^2 + 2gZ\Pi\}d\Pi d\Theta^* dZ.$$

Now consider the quadric,

$$ax^2 + by^2 + cz^2 + 2gzx = 1;$$

when referred to its principal axes this becomes

$$\sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 z^2 = 1.$$

Further, the discriminant of the quadric is unaltered by the transformation, so that the coefficients satisfy the relation,

$$\sigma_1^2 \sigma_2^2 \sigma_3^2 \equiv \Delta \equiv \begin{vmatrix} a & 0 & g \\ 0 & b & 0 \\ g & 0 & c \end{vmatrix}.$$

The velocity-ellipsoid is now referred to its principal axes, and the expression for the particle-density becomes

$$\nu = \iiint_{u,\,v,\,w=-\infty}^{+\infty} f\{(\phi - \omega^2 R^2/4b) + \sigma_1^2 u^2 + \sigma_2^2 v^2 + \sigma_3^2 w^2\} du \,.\, dv \,.\, dw.$$

For the final transformation it is convenient to put

$$\sigma_1 u = V \sin\theta \cos\psi, \qquad \sigma_2 v = V \sin\theta \sin\psi, \qquad \sigma_3 w = V \cos\theta.$$

The integrations with respect to θ and ψ can now be performed, so that

$$\nu = \frac{4\pi}{\Delta^{\frac{1}{2}}} \int_{V=0}^{\infty} f\{V^2 + \phi - \omega^2 R^2/4b\} V^2 dV.$$

Finally, we define the functions, ϕ^* and g, such that

$$\phi^* = \phi - \omega^2 R^2/4b, \qquad g(p^2) = \int_{q=0}^{\infty} f(p^2 + q^2) q^2 dq.$$

Then the final expression for the particle-density is

$$\nu = \frac{4\pi}{\Delta^{\frac{1}{2}}} g(\phi^*).$$

As the function f has never been specified, so the function g remains unknown, but further progress can be made if it is assumed that the particles themselves provide the entire gravitational field, and that all the particles are of the same mass. In this case the ratio of the mass-density to the particle-density is simply the mass of a star.

7. *There are no Self-Gravitating Systems of Stars of Equal Mass which satisfy all the General Conditions.*—In order to find a pair of functions, F and G, which will represent a self-gravitating system of stars of equal mass, the ratio of the mass-density to the particle-density is now made constant. It is recognized, of course, that the galactic system does not satisfy this condition, for the stars are by no means of the same mass. Further, the spatial distribution of stars of different type, and particularly such features as the concentration of cepheid variables and planetary nebulæ near the galactic plane, suggest that the average mass of a star is not the same at all points of the system. Nevertheless, there remains the mathematical problem of the distribution of stars of equal mass, a problem which should at least have some bearing on the structure of the galaxy. It is rather surprising then to find that the assumptions which have been made are sufficiently restrictive to prevent there being any such mathematical solution.

In the general case, with the variables ξ and η, the equation between the mass-density and the particle-density is

$$\nabla^2 \Omega = \frac{m}{\Delta^{\frac{1}{2}}} g(\phi^*).$$

The function g is unknown, but can be eliminated by multiplying through by $\Delta^{\frac{1}{2}}$ and then differentiating with respect to ξ, and also with respect to η, to form the two equations:

$$\Delta^{\frac{1}{2}}\frac{\partial}{\partial\xi}(\nabla^2\Omega)+\tfrac{1}{2}\Delta^{-\frac{1}{2}}\frac{\partial\Delta}{\partial\xi}\,.\,\nabla^2\Omega=mg'\frac{\partial\phi^*}{\partial\xi},$$

$$\Delta^{\frac{1}{2}}\frac{\partial}{\partial\eta}(\nabla^2\Omega)+\tfrac{1}{2}\Delta^{-\frac{1}{2}}\frac{\partial\Delta}{\partial\eta}\,.\,\nabla^2\Omega=mg'\frac{\partial\phi^*}{\partial\eta}.$$

Now, by cross-multiplication, g' is eliminated, and there remains a single equation in two variables, namely,

$$\frac{\partial\phi^*}{\partial\eta}\left\{2\Delta\frac{\partial}{\partial\xi}(\nabla^2\Omega)+\nabla^2\Omega\,.\,\frac{\partial\Delta}{\partial\xi}\right\}=\frac{\partial\phi^*}{\partial\xi}\left\{2\Delta\frac{\partial}{\partial\eta}(\nabla^2\Omega)+\nabla^2\Omega\,.\,\frac{\partial\Delta}{\partial\eta}\right\}. \tag{26}$$

This equation is rather too heavy for immediate treatment, but a considerable reduction is effected if new quantities, W, U_1, V_1, U_2, V_2, are defined by the following equations :—

$$\begin{aligned}W&=(\xi-\eta)^3\,.\,\nabla^2\Omega,\\&=6(\xi F'+\eta G')(\xi-\eta)-2\beta(F'+G')(\xi-\eta)+2(F-G)(\xi+\eta+2\beta)\\&\quad-8F'(\xi^2-\beta^2)+8G'(\eta^2-\beta^2)+4F''(\xi^2-\beta^2)(\xi-\eta)+4G''(\eta^2-\beta^2)(\xi-\eta),\end{aligned}$$

$$\begin{aligned}U_1&=(\xi-\eta)^4\frac{\partial}{\partial\xi}(\nabla^2\Omega),\\&=6(F'+\xi F'')(\xi-\eta)^2-2\beta F''(\xi-\eta)^2+2(F-G)(\xi-\eta)\\&\quad+2F'(\xi+\eta+2\beta)(\xi-\eta)-16F''(\xi^2-\beta^2)(\xi-\eta)-16F'\xi(\xi-\eta)\\&\quad+4F'''(\xi^2-\beta^2)(\xi-\eta)^2+8F''\xi(\xi-\eta)^2-12(\xi F'+\eta G')(\xi-\eta)\\&\quad+4\beta(F'+G')(\xi-\eta)-6(F-G)(\xi+\eta+2\beta)+24F'(\xi^2-\beta^2)\\&\quad-24G'(\eta^2-\beta^2)-8G''(\eta^2-\beta^2)(\xi-\eta),\end{aligned}$$

$$\begin{aligned}V_1&=(\xi-\eta)^4\frac{\partial}{\partial\eta}(\nabla^2\Omega),\\&=6(G'+\eta G'')(\xi-\eta)^2-2\beta G''(\xi-\eta)^2+2(F-G)(\xi-\eta)-2G'(\xi+\eta+2\beta)(\xi-\eta)\\&\quad+16G''(\eta^2-\beta^2)(\xi-\eta)+16G'\eta(\xi-\eta)+4G'''(\eta^2-\beta^2)(\xi-\eta)^2\\&\quad+8G''\eta(\xi-\eta)^2+12(\xi F'+\eta G')(\xi-\eta)-4\beta(F'+G')(\xi-\eta)\\&\quad+6(F-G)(\xi+\eta+2\beta)-24F'(\xi^2-\beta^2)+24G'(\eta^2-\beta^2)+8F''(\xi^2-\beta^2)(\xi-\eta),\end{aligned}$$

$$\begin{aligned}U_2&=(2\beta b)^2(\xi-\eta)^2\frac{\partial\phi^*}{\partial\xi}\Big/(a_3\eta+\gamma)\\&=2(2\beta b)^2\{F-G-F'(\xi-\eta)\}-\tfrac{1}{2}\omega^2\beta(\eta+\beta)(\xi-\eta)^2,\end{aligned}$$

$$\begin{aligned}V_2&=(2\beta b)^2(\xi-\eta)^2\frac{\partial\phi^*}{\partial\eta}\Big/(a_3\xi+\gamma),\\&=2(2\beta b)^2\{G-F-G'(\eta-\xi)\}-\tfrac{1}{2}\omega^2\beta(\xi+\beta)(\xi-\eta)^2.\end{aligned}$$

Further, it should be noted that

$$\begin{aligned}\Delta&=b(ac-g^2)=b\{a_0a_4+a_3(a_4z^2+a_0R^2)\}=b(a_3\xi+\gamma)(a_3\eta+\gamma),\\2\beta b&=(a_2-a_3)\xi\eta+(a_2+a_3)\beta(\xi+\eta)+(a_2-a_3)\beta^2+2\beta a_0.\end{aligned}$$

It is then seen that (26) is divisible by the factor,

$$(a_3\xi+\gamma)(a_3\eta+\gamma).$$

With this factor removed, the equation (26) is reduced to the form,

$$V_2\left\{2b(\alpha_3\xi+\gamma)U_1+\alpha_3 b(\xi-\eta)W+(\alpha_3\xi+\gamma)\frac{\partial b}{\partial\xi}(\xi-\eta)W\right\}$$
$$=U_2\left\{2b(\alpha_3\eta+\gamma)V_1+\alpha_3 b(\xi-\eta)W+(\alpha_3\eta+\gamma)\frac{\partial b}{\partial\eta}(\xi-\eta)W\right\}. \quad (27)$$

To make any further progress with this, it is found most convenient to expand F as a function of ξ in descending powers. The justification for this step depends on the assumption that the system is in equilibrium under its own gravitational attractions only. In this case the gravitational potential at a great distance r from the mass-centre of the system is of the order of M/r, where M is a constant, proportional to the finite mass of the system. Now it has been shown that for large values of r, η is approximately $\beta\cos 2\chi$, and ξ is equal to r^2; hence Ω is of the order of $M/\xi^{\frac{1}{2}}$, or $F(\xi)$ is of the order of $\xi^{\frac{1}{2}}$. It may reasonably be supposed that Ω can be expanded in descending integral powers of r, or in descending integral and half-integral powers of ξ, but this further assumption is not found to be necessary.

It will first be assumed that F is expanded in the form

$$F(\xi)=M\xi^{\frac{1}{2}}+a_c\xi^c+\text{smaller terms.}\quad(0<c<\tfrac{1}{2}.)$$

This is substituted in (27), and the dominant terms in ξ in each bracket are selected. The principal terms with the coefficient M in W, U_1, and V_1 all vanish identically, and the leading terms in these quantities are respectively

$$2a_c(2c-1)(c-1)\xi^{c+1},$$
$$2a_c(2c-1)(c-1)(c-2)\xi^{c+1},$$
$$\{4G'''(\eta^2-\beta^2)+14G''\eta-2G''\beta+4G'\}\xi^2.$$

In the same way it will be seen that the dominant term of U_2 is only of the order $\xi^{\frac{5}{2}}$, but the greatest term in V_2 is

$$[2G'\{(\alpha_2-\alpha_3)\eta+(\alpha_2+\alpha_3)\beta\}^2-\tfrac{1}{2}\omega^2\beta]\xi^3.$$

When all these expressions are substituted in (27), the left-hand side is of order $(6+c)$ in ξ, but on the right-hand side the greatest term is only of order $(5\frac{1}{2}+c)$. It follows that the coefficient of the greatest term on the left-hand side must vanish. Thus, either $a_c=0$, or else

$$G'\{(\alpha_2-\alpha_3)\eta+(\alpha_2+\alpha_3)\beta\}^2=\tfrac{1}{4}\omega^2\beta. \quad (28)$$

The first possibility suggests the expansion should be

$$F(\xi)=M\xi^{\frac{1}{2}}+a_0+b_c\xi^{-c}+\text{smaller terms.}\quad(c>0.)$$

Again the dominant term is on the left-hand side, and the condition that it should vanish is that, either (28) is satisfied or else that G must satisfy the equation,

$$4G''(\eta^2-\beta^2)+6G'\eta-2G'\beta-2(G-a_0)=0. \quad (29)$$

The first step in examining (28) is to integrate to obtain the expression for G, namely,

$$G=G_0-\tfrac{1}{4}\omega^2\beta/\{(\alpha_2-\alpha_3)\eta+(\alpha_2+\alpha_3)\beta\}(\alpha_2-\alpha_3).$$

This is substituted in (27) and that equation is then multiplied through by $\{(\alpha_2-\alpha_3)\eta+(\alpha_2+\alpha_3)\beta\}^5$, so that it becomes just a polynomial in η. If the coefficients of the different powers of η are equated to zero, a series of equations in ξ is obtained. From the highest power of η, namely η^{11}, occurring only on the right-hand side, we must have either

$$2\{(\alpha_2-\alpha_3)\xi+(\alpha_2+\alpha_3)\beta\}^2F'-\tfrac{1}{2}\omega^2\beta=0$$

or else

$$\{(a_2-a_3)\xi+(a_2+a_3)\beta\}2a_3\{4F''(\xi^2-\beta^2)+6F'\xi-2F'\beta-2(F-G_0)\}=0.$$

The first alternative is impossible, since it integrates to give

$$F=F_0-\tfrac{1}{4}\omega^2\beta/\{(a_2-a_3)\xi+(a_2+a_3)\beta\}(a_2-a_3);$$

this is contrary to the requirement that $F(\xi)\sim M\xi^{\frac{1}{2}}$. It therefore follows that F must satisfy the equation,

$$2F''(\xi^2-\beta^2)+3F'\xi-F'\beta-(F-G_0)=0. \tag{30}$$

This can be integrated once without difficulty to give

$$2F'(\xi^2-\beta^2)-(F-G_0)(\xi+\beta)=A. \tag{31}$$

The equations (30), (31), and the derivative of (30), are now substituted in the expressions for W, U_1 and V_1, which become

$$W=\{4G''(\eta^2-\beta^2)+6G'\eta-2G'\beta-2(G-G_0)\}(\xi-\eta)+8G'(\eta^2-\beta^2)$$
$$-4(G-G_0)(\eta+\beta)-4A,$$

$$U_1=-\{8G''(\eta^2-\beta^2)+12G'\eta-4G'\beta-4(G-G_0)\}(\xi-\eta)-24G'(\eta^2-\beta^2)$$
$$+12(G-G_0)(\eta+\beta)+12A,$$

$$V_1=\{4G'''(\eta^2-\beta^2)+14G''\eta-2G''\beta+4G'\}(\xi-\eta)^2+\{16G''(\eta^2-\beta^2)$$
$$+24G'\eta-8G'\beta-8(G-G_0)\}(\xi-\eta)+24G'(\eta^2-\beta^2)-12(G-G_0)(\eta+\beta)-12A.$$

In these expressions the values of G and its derivatives are substituted from (28). If δ denotes the ratio $(a_2+a_3)/(a_2-a_3)$, it is found that

$$2G'(\eta^2-\beta^2)+(G-G_0)(\eta+\beta)\equiv\frac{\omega^2\beta}{4(a_2-a_3)^2}\left[3+\frac{\beta-5\delta}{\eta+\delta}-2\frac{\beta^2-\delta^2}{(\eta+\delta)^2}\right].$$

Hence, by differentiation, it follows that

$$2G''(\eta^2-\beta^2)+3G'\eta-G'\beta-(G-G_0)\equiv\frac{\omega^2\beta}{4(a_2-a_3)^2}\left[-\frac{\beta-5\delta}{(\eta+\delta)^2}+4\frac{\beta^2-\delta^2}{(\eta+\delta)^3}\right],$$

and, by another differentiation, that

$$2G'''(\eta^2-\beta^2)+7G''\eta-G''\beta+2G'\equiv\frac{\omega^2\beta}{4(a_2-a_3)^2}\left[2\frac{\beta-5\delta}{(\eta+\delta)^3}-12\frac{\beta^2-\delta^2}{(\eta+\delta)^4}\right].$$

Thus the three expressions W, U_1, V_1 finally become

$$W=\frac{\omega^2\beta}{4(a_2-a_3)^2}\left[(\xi-\eta)\left\{-2\frac{\beta-5\delta}{(\eta+\delta)^2}+8\frac{\beta^2-\delta^2}{(\eta+\delta)^3}\right\}+12+4\frac{\beta-5\delta}{\eta+\delta}-8\frac{\beta^2-\delta^2}{(\eta+\delta)^2}\right]-4A,$$

$$U_1=\frac{\omega^2\beta}{4(a_2-a_3)^2}\left[(\xi-\eta)\left\{4\frac{\beta-5\delta}{(\eta+\delta)^2}-16\frac{\beta^2-\delta^2}{(\eta+\delta)^3}\right\}-36-12\frac{\beta-5\delta}{\eta+\delta}+24\frac{\beta^2-\delta^2}{(\eta+\delta)^2}\right]+12A,$$

$$V_1=\frac{\omega^2\beta}{4(a_2-a_3)^2}\left[(\xi-\eta)^2\left\{4\frac{\beta-5\delta}{(\eta+\delta)^3}-24\frac{\beta^2-\delta^2}{(\eta+\delta)^4}\right\}-(\xi-\eta)\left\{8\frac{\beta-5\delta}{(\eta+\delta)^2}-32\frac{\beta^2-\delta^2}{(\eta+\delta)^3}\right\}\right.$$
$$\left.+36+12\frac{\beta-5\delta}{\eta+\delta}-24\frac{\beta^2-\delta^2}{(\eta+\delta)^2}\right]-12A.$$

Equation (27) is now multiplied by $(\eta+\delta)^5$, and the expressions for W, U_1, V_1 are substituted. The new equation is a polynomial of the tenth degree in η. The coefficient of η^{10} must vanish, so that

$$\{2(a_2-a_3)^2(\xi+\delta)^2F'-\tfrac{1}{2}\omega^2\beta\}2a_3(\xi+\delta)\left\{24\frac{\omega^2\beta}{4(a_2-a_3)^2}-8A\right\}=0.$$

Hence it follows that $A=3\omega^2\beta/4(\alpha_2-\alpha_3)^2$. When A has this value, the quantities W, U_1, V_1, multiplied by $(\eta+\delta)^5$, are now all lower by one degree in η. Hence the highest power of η still occurs only on the right-hand side. Its coefficient is

$$\frac{\omega^2\beta}{4(\alpha_2-\alpha_3)^2}(\beta-5\delta)\{2(\alpha_2-\alpha_3)^2(\xi+\delta)^2F'-\tfrac{1}{2}\omega^2\beta\}36a_3(\xi+\delta).$$

This coefficient can only vanish if $\beta=5\delta$. But, from the definition of δ, this gives $(\alpha_2-\alpha_3)=5(\alpha_2+\alpha_3)$, or $2a_2+3a_3=0$. This result is inadmissible, for it is expressly required that the axes of the velocity-ellipsoid should be positive definite for all values of the space co-ordinates. Unless this restriction is imposed, it is possible to have a finite number of bodies with infinite velocities within the system; if the restriction is imposed, it follows that both α_2 and α_3 must be positive. Thus the equation $\beta=5\delta$ cannot be satisfied, and there exists no general solution of the problem which will satisfy (28).

There is one exceptional case which should not be overlooked, namely, that in which the two constants α_2 and α_3 are equal. From (28) this gives that G' is constant. There is no need to give the details of this solution. When the expression for G is substituted in (27), that equation becomes a polynomial of the seventh degree in η, and, as before, the coefficients of the separate powers are equated to zero. Again the solution turns out to be an impossible one.

As there is no solution of the problem when equation (28) is used, it remains to investigate the only alternative equation, (29), for it was shown that a solution of the general problem must satisfy either (28) or (29).

The treatment of (29) is rather different; the equation can be integrated once to give

$$2G'(\eta^2-\beta^2)-(G-a_0)(\eta+\beta)=B. \tag{32}$$

It should be noticed that the equations (29) and (32) which G must now satisfy are exactly the same as (30) and (31), previously satisfied by F; it can therefore be deduced that the quantities W, U_1, V_1 will now be independent of G, and will only contain η to the first power in W and V_1, and to the second power in U_1.

The integration of (32) takes one of two forms, according as β is positive or negative. The actual method is the same in either case, so only the solution when β is positive will be given. It is

$$G=a_0+C(\beta-\eta)^{\frac{1}{2}}-B\left\{1-\sqrt{\frac{\beta-\eta}{2\beta}}\tanh^{-1}\sqrt{\frac{\beta-\eta}{2\beta}}\right\}\Big/2\beta. \tag{33}$$

Since the terms W, U_1, V_1 are now independent of G, the hyperbolic tangent which occurs in (33) will only occur in the factors U_2 and V_2 when the values of G and its derivatives are substituted in (27). So, by sorting out the factors which contain the hyperbolic tangent and those which are independent of it, we can write (27) in the form

$$P(\xi,\eta)\tanh^{-1}\sqrt{\frac{\beta-\eta}{2\beta}}+Q(\xi,\eta)=0. \tag{34}$$

In this, P and Q are functions of ξ and η, but η is only contained in the form of integral powers of the quantity $(\beta-\eta)^{\frac{1}{2}}$. Thus, as P and Q are both polynomials in $(\beta-\eta)^{\frac{1}{2}}$, (34) can only be satisfied by having P and Q to be identically zero. Since the coefficient of the hyperbolic tangent must vanish, it follows that

$$8\beta^2b^2B(2\beta)^{-\frac{3}{2}}(\beta-\eta)^{\frac{1}{2}}\left[\left(1-\tfrac{1}{2}\frac{\xi-\eta}{\beta-\eta}\right)\left\{2b(a_3\xi+\gamma)U_1+a_3b(\xi-\eta)W+(a_3\xi+\gamma)\frac{\partial b}{\partial\xi}(\xi-\eta)W\right\}\right.$$
$$\left.+\left\{2b(a_3\eta+\gamma)V_1+a_3b(\xi-\eta)W+(a_3\eta+\gamma)\frac{\partial b}{\partial\eta}(\xi-\eta)W\right\}\right]=0. \tag{35}$$

It is supposed that B is not zero, and so this and other non-zero factors can be removed, to leave a polynomial of the fourth degree in η, which must vanish identically. The coefficient of η^4 must be zero, so that F must satisfy the equation,

$$2(\alpha_3\xi+\gamma)(\xi+\delta)\{2F'''(\xi^2-\beta^2)+7F''\xi-F''\beta+2F'\}$$
$$=\{3\alpha_3(\xi+\delta)+(\alpha_3\xi+\gamma)\}\{2F''(\xi^2-\beta^2)+3F'\xi-F'\beta-(F-a_0)\}.$$

This can be integrated once, to give

$$2F''(\xi^2-\beta^2)+3F'\xi-F'\beta-(F-a_0)=D(\alpha_3\xi+\gamma)^{\frac{5}{2}}(\xi+\delta)^{-\frac{1}{2}}.$$

But it is known that for large values of ξ, F is of the order $(M\xi^{\frac{1}{2}}+a_0)$, and consequently, as $\xi\to\infty$,

$$2F''(\xi^2-\beta^2)+3F'\xi-F'\beta-(F-a_0)\to 0.$$

It therefore follows that D must be zero, and so F satisfies equations (30) and (31) again. The expressions for W, U_1, V_1 are now respectively $4(B-A)$, $-12(B-A)$, $12(B-A)$. It is clearly impossible for A and B to be equal, for then W is zero, and, from the definition of W, the mass-density is zero at all points. The factor $(B-A)$ can therefore be removed from (35), which now becomes

$$(3\eta-\xi-2\beta)\left\{-6b(\alpha_3\xi+\gamma)+\alpha_3 b(\xi-\eta)+(\alpha_3\xi+\gamma)\frac{\partial b}{\partial\xi}(\xi-\eta)\right\}$$
$$+2(\eta-\beta)\left\{6b(\alpha_3\eta+\gamma)+\alpha_3 b(\xi-\eta)+(\alpha_3\eta+\gamma)\frac{\partial b}{\partial\eta}(\xi-\eta)\right\}=0.$$

This obviously cannot be satisfied by any values of the constants except for the trivial case of zero values for all.

There are two exceptions to the above proof that no general system can be found for which the function G satisfies (29); the first is the case in which the constant B is identically zero, for then there is no hyperbolic tangent introduced into the solution of (29) for G, which, in fact, reduces to the form

$$G=a_0+C(\beta-\eta)^{\frac{1}{2}}. \tag{36}$$

Again G vanishes from the terms W, U_1, V_1, and the surd $(\beta-\eta)^{\frac{1}{2}}$ only occurs, always with the factor C, in U_2 and V_2. When the expression for G is substituted in (27), the terms containing the surd can be separated, so that (27) is written

$$P_1(\xi,\ \eta)C(\beta-\eta)^{\frac{1}{2}}+Q_1(\xi,\ \eta)=0.$$

But it will be seen from a comparison of (33) and (36) that the coefficient of the constant C will be exactly P, the coefficient of the hyperbolic term when B was not zero. Hence when the terms containing the surd are separately equated to zero, it is required that the same function P shall vanish, and again there are no values of the constants which can accomplish this.

There still remains the final exceptional case, when B and C are both zero. In this case G is simply a constant, and (27) becomes a polynomial of the sixth degree in η. Again the final solution is found to be an impossible one.

This concludes the treatment of the general solution—that is, the case in which the velocity-ellipsoid has three unequal axes, even at the centre. For the slightly restricted problem, when the two constants, a_0 and a_4, are equal, the simpler solutions for Ω and ϕ, given by (23), are used. The general method is exactly the same as has been used already in this section, and the conclusion is the same, that no system of the type considered can be in equilibrium under its own gravitational attraction only.

The final possibility to be investigated is the case given by Oort, namely, the system in which the axes of Π and Z are always equal. Although the galactic system does not possess this property, the investigation is necessary, for the system with two equal axes would appear to be quite a possible one.

8. *The Special Case of Systems with Two Equal Axes.*—It now remains to examine the possibility of self-gravitation in systems possessing velocity-ellipsoids with two equal axes. This is the type given by Oort, in which the coefficients in the velocity-ellipsoid are

$$a=c=a_0, \qquad b=a_0+R^2a_2, \qquad f=g=h=0.$$

Further, the function ϕ is given by

$$\phi=C-2a_0\Omega.$$

Hence the particle-density of the system is expressed as

$$\nu=4\pi\,.\,g(C-2a_0\Omega-\omega^2R^2/4b)/a_0(a_0+R^2a_2)^{\frac{1}{2}}.$$

The mass-density is given in terms of the unknown gravitational potential by the equation,

$$\rho=-\nabla^2\Omega=-\left[\frac{\partial^2\Omega}{\partial r^2}+\frac{2}{r}\frac{\partial\Omega}{\partial r}+\frac{1}{r^2}\left(\frac{\partial^2\Omega}{\partial\chi^2}-\frac{\partial\Omega}{\partial\chi}\tan\chi\right)\right].$$

The equation expressing the condition that the mass-density and the particle-density are proportional can be written

$$\rho(a_0+R^2a_2)^{\frac{1}{2}}=mg(C-2a_0\Omega-\omega^2R^2/4b).$$

Here, as before, the unknown function, g, is eliminated by differentiating with respect to r and also with respect to χ and then cross-multiplying. In this way we obtain the equation,

$$\left\{(a_0+a_2r^2\cos^2\chi)\frac{\partial\rho}{\partial\chi}-a_2r^2\cos\chi\sin\chi\,.\,\rho\right\}\left\{b^2\frac{\partial\Omega}{\partial r}+\tfrac{1}{4}\omega^2r\cos^2\chi\right\}$$
$$=\left\{(a_0+a_2r^2\cos^2\chi)\frac{\partial\rho}{\partial r}+a_2r\cos^2\chi\,.\,\rho\right\}\left\{b^2\frac{\partial\Omega}{\partial\chi}-\tfrac{1}{4}\omega^2r^2\cos\chi\sin\chi\right\}. \qquad (37)$$

As in the previous cases, it is now supposed that the gravitational potential can be expanded in a descending power series in r, namely,

$$\Omega=Mr^{-1}+S_c(\chi)r^{-c}+\text{smaller terms.} \qquad (c>1.)$$

In this, M is a constant, but S_c is, in general, a function of χ. The mass-density will therefore approximately be

$$\rho=-\{S_c''-S_c'\tan\chi+c(c-1)S_c\}r^{-(c+2)}+\text{smaller terms.}$$

The expressions for Ω and ρ are now substituted in (37), and the dominant terms are collected. On the left-hand side the highest power of r is $(2-c)$, but on the right-hand side it is either $(1-c)$ or $(3-2c)$, whichever is the larger; as $c>1$, the left-hand side is dominant, and the coefficient of the power $(2-c)$ must vanish. This gives the equation:

$$Ma_2^2\cos^4\chi\left[\cos^2\chi\frac{\partial}{\partial\chi}\{S_c''-S_c'\tan\chi+c(c-1)S_c\}\right.$$
$$\left.-\cos\chi\sin\chi\{S_c''-S_c'\tan\chi+c(c-1)S_c\}\right]=0.$$

This can be integrated, and gives

$$S_c''-S_c'\tan\chi+c(c-1)S_c=A\sec\chi. \qquad (38)$$

But at points on the galactic axis, χ is 90° and sec χ is infinite. Hence, unless $A=0$, the term which is dominant in the density is infinite at all points on the galactic axis. A system in which the density is infinite at all points of a finite line is of no physical interest, and it must therefore be taken that $A=0$. It now follows that S_c satisfies the equation,

$$S_c'' - S_c' \tan \chi + c(c-1)S_c = 0. \tag{39}$$

This in turn implies that the density falls more rapidly than $r^{-(c+2)}$. The gravitational potential is therefore expanded to further terms, namely,

$$\Omega = Mr^{-1} + S_c r^{-c} + S_d r^{-d} + \ldots + S_k r^{-k} + \ldots$$

The indices $c, d, \ldots k, \ldots$ form an increasing sequence, and, for all indices less than k, the corresponding coefficient, S, satisfies an equation of the type (39). However, the coefficient S_k is such that

$$S_k'' - S_k' \tan \chi + k(k-1)S_k \neq 0. \tag{40}$$

Then the first approximation to the mass-density is

$$\rho = -\{S_k'' - S_k' \tan \chi + k(k-1)S_k\} r^{-(k+2)}.$$

The expressions for Ω and ρ are now substituted in (37) and the leading terms collected. The first bracket on the left-hand side is of the order r^{-k}, and the second is approximately $M{a_2}^2 \cos^4 \chi \,.\, r^2$. On the right-hand side the two brackets are of the order $r^{-(k+1)}$ and r^2 or $r^{(4-c)}$ respectively, so that the left-hand side is still dominant, and, from the vanishing of the coefficient of the highest power, it is found that S_k must satisfy an equation of the type (38). Again the constant A must be zero, or else the density becomes infinite along the galactic axis, and if A is zero, then S_k cannot satisfy the relation (40).

There would appear to be one way out of the difficulty, namely, it might be supposed that all the coefficients S satisfy equations of the type (39), so that there is no index k. But in this case the mass-density is zero at all points, and again the solution is an artificial one.

One special case remains to be examined—the case in which a_2 is identically zero. Equation (37) now takes the form

$$\frac{\partial \rho}{\partial \chi}\left\{ {a_0}^2 \frac{\partial \Omega}{\partial r} + \tfrac{1}{4}\omega^2 r \cos^2 \chi \right\} = \frac{\partial \rho}{\partial r}\left\{ {a_0}^2 \frac{\partial \Omega}{\partial \chi} - \tfrac{1}{4}\omega^2 r^2 \cos \chi \sin \chi \right\}.$$

The terms containing ω are now dominant, and from them it is seen that, to a first approximation, the density is a function of $r \cos \chi$ only. Here again there is no solution of physical interest, and it is therefore concluded that there can be no system which possesses a generalized ellipsoidal distribution of velocities, which is in a steady state, and which is maintained only by its own gravitational attractions.

9. *The Assumptions, Direct and Implicit, and the Conclusions deduced from them.*—The principal assumptions which this paper sets out to test are the possible existence of a stellar system with a generalized ellipsoidal distribution of velocities of the type (3), and, further, the steady state of such a system. It is not necessary that the system should possess the properties of the galactic system, but it must possess certain reasonable physical properties, such as finite density and finite gravitational potential—except possibly a singularity at the centre—and it is also required that infinite velocities should be infinitely improbable.

It is supposed that the system is finite in extent and that the gravitational potential at a great distance r from the centre of the system is of the order of M/r, where M is a constant. This may not be realized in the universe of nebulæ, for the existence of so many double systems suggests that the gravitational field of one galaxy will be quite

effective in an adjacent galaxy. In the earlier part of the paper—in the first six sections —it is usually possible to manage without this assumption, by considering equipotential surfaces and similar devices.

There are two further restrictions imposed on the system. The gravitational potential is not spherically symmetrical, and the mean motion of the system is not zero at every point. These assumptions are made, of course, because the galactic system and the extra-galactic nebulæ appear to satisfy them both, and because the second assumption in particular leads to great simplification in the subsequent work. It is probably unnecessary to assume that the distribution is not spherical, for it could be reasoned that if the system is in a steady state, and if the mean motion is not zero, then the system must be rotating, and so will tend to be flattened.

With these assumptions it is then shown, first, that the gravitational potential possesses rotational symmetry, and then, that the entire velocity-distribution has such symmetry.

The next step is the evaluation of the gravitational potential necessary to maintain the system; this is done in the completely general case, and also in the two special cases, when the constants of integration take special values, thus causing the general solution to fail. This process involves no fresh assumptions, and determines the potential as a special combination of two arbitrary functions, each of a single variable. One of the functions controls the variation of the gravitational potential with distance, and the other determines the extent of the flattening.

The mass-density at all points of the system is determined from the gravitational potential by means of Poisson's equation. This implies a further assumption, since Poisson's equation is based on the Newtonian law of gravitation. However, after the use of the Newtonian laws of motion at the very outset, it would be pointless at this stage to try to replace the Newtonian law of gravitation by a relativistic equivalent.

The determination of the particle-density by integration of the velocity-distribution function involves no new conceptions.

The remaining sections of the paper are concerned with a more restricted problem, namely, that in which the system of particles has a velocity-distribution which satisfies the requirements already set out in this section, but now it is supposed that the only forces acting on the system are the gravitational attractions of the particles themselves. In this case the mass-density, derived from the potential, and the particle-density, obtained by integration, must be proportional.

This restriction to the consideration of self-gravitating systems only, marks the point at which theory and observation most clearly part company, for it can be deduced from such features as the concentration of certain types of object towards the galactic plane that different types of object may have the same *type* of velocity-distribution, but they do not appear to have exactly the same distribution. Thus, if the distribution-function is integrated, it gives only the particle-density for a particular type of object, and, in general, this will only form a part of the mass-density, since there will be objects of other types in the same element of space. Yet the problem of systems of identical objects such as is here considered is of importance, for it would seem unreasonable if a system of stars of equal mass could not be in equilibrium, although another system, with the same type of velocity-distribution, could be in equilibrium solely because it consisted of stars of unequal masses.

In the treatment of this problem it was found convenient to assume that the gravitational potential could be expanded in series. This is quite possible at great distances from the centre of an isolated system, since it is based on the Newtonian potential, expanded in binomial series, and the expansion is only valid for large values of the distance. Again, therefore, it is necessary to suppose that the system is not acted upon by neighbouring systems. This may only be a difficulty of the method of procedure, since the constant ratio of mass- and particle-density is not in itself inconsistent with the

existence of external galaxies, each with their contribution to the gravitational field but not to the mass-density. Indeed, it should be possible to solve (26) or (27) for F and G without assuming that F can be so expanded.

The conclusion to this part of the investigation is that there can be no system which has the ellipsoidal distribution, which is in a steady state, and which is maintained solely by its own gravitational attraction.

Since it is found that the system here examined cannot permanently exist, the next step clearly is the withdrawal of the assumption that the system is in a steady state, and a certain amount of progress has already been made on these lines. Unfortunately, whereas the problem of the system in the steady state has in effect only two space co-ordinates—for it has been shown that the whole distribution is independent of the longitude—the problem in the case of the non-steady state has three space co-ordinates, since the existence of rotational symmetry can no longer be proved, and the time itself provides a fourth co-ordinate. However, the complete solution of the steady problem should be of great assistance in dealing with ellipsoidal systems which are not in a steady state.

Summary

The dynamical features of a stellar system possessing a generalized ellipsoidal distribution of velocities are investigated. The velocity-distribution has been generalized in the sense that, unlike Schwarzschild's ellipsoidal law, it is not associated with the exponential function, but requires that the distribution is a function of a single quantity, which, in turn, involves the velocity-components only as the sum of linear and quadratic terms. The problem is also restricted to the case of systems in a steady state. Further, it is assumed that the mean motion of the system is not zero at all points.

Under these conditions it is shown that the gravitational potential which controls the system must have rotational symmetry about an axis. Also, apart from the possible exception of systems which are spherically symmetrical, the entire velocity-distribution is symmetrical about that axis. The three axes of the velocity-ellipsoid will, in general, be unequal, but two of them always lie in a plane through the axis of symmetry, so that there is no deviation in longitude between the vertex of star-streaming and the direction of the galactic centre.

The gravitational potential has to satisfy a partial differential equation; the general solution of this, and also one special solution, are given for the first time.

Finally, the problem is restricted still further, and an attempt is made to find such a system of stars of equal mass which is entirely self-gravitating. For such a system the mass-density and the particle-density are in a constant ratio. The particle-density is obtained by integration of the distribution-function, and the mass-density can be evaluated by applying the Laplacian operator to the gravitational potential. However, it is established that there can be no self-gravitating system with a generalized ellipsoidal distribution of velocities which is in a steady state.

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Additional Note

After the above paper had been completed, the author had occasion to refer to Eddington's fundamental paper on this subject and was rather dismayed to find that the general solution of (22), which he had believed was published here for the first time, had in fact been given by Eddington twenty-five years ago.* It is most remarkable that (22) was first published in its present form twelve years ago, and yet it has never before been noticed that the solution had already been found.

Chandrasekhar † has pointed out that Eddington's treatment was not completely general, since the notion of principal velocity surfaces is a severe restriction. It is therefore of interest to notice that the system with complete rotational symmetry is actually of Eddington's type; the principal velocity surfaces are confocal spheroids and hyperboloids of revolution, determined by the parameters, ξ and η, of section 4.

The present paper is an advance on Eddington's in three respects: the conception of velocity surfaces has not been needed, the existence of rotational symmetry has been established, and the treatment is not confined to the original exponential form of velocity-distribution.

* *M.N.*, 76, 46, 1915; E. T. Whittaker, *Analytical Dynamics*, p. 320, Cambridge, 1904.
† *Ap. J.*, 90, 43, 1939.

ON THE ORIGIN OF THE SOLAR SYSTEM

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Introduction.—Since the publication of the binary star hypothesis of the origin of the solar system there appears to have been a certain amount of misgiving about the scientific acceptability of theories of the encounter type. This seems to have arisen owing to the circumstance that the initial conditions required to enable a suitable encounter to take place would not be likely to occur frequently in any ordinary distribution of stars. The same is also true of course for the original tidal and collisional hypotheses wherein the Sun is regarded as one of the parent stars. Now any theory of the origin of the solar system is necessarily a theory of the past and therefore its initial hypotheses cannot be subjected to any direct experimental test. In other words, no decision can be made as to whether or not, given a consistent theory of the origin of the solar system, our planetary system came into existence in the manner described by the theory; the theory can only show that the system could have been formed in a certain way. On the other hand, with regard to any particular theory of the origin of the solar system, the tests to which it can at present be subjected are those that concern its dynamical and physical validity. There is, however, no reason to suppose that only a single tenable theory can exist. Indeed, from the nature of the problem it is evident that appropriate initial conditions will always occupy a whole range of possible values (of velocities, positions, etc.), though such range may be limited, and it is only a stage further from this to inquire whether any essentially different motion or process could result in the formation of a system of planets.

We are therefore led to consider the possibility of alternative initial hypotheses. The present form of the collision theory leads to the view that solar systems formed in accordance with it are probably of very infrequent occurrence and in any general account of stellar evolution are to be regarded as exceptional objects rather than as representing a stage through which any or even a small fraction of stars must pass during their evolution. This may be contrasted with Laplace's nebular hypothesis in which the solar system was ascribed to a process of stellar evolution, though at the time this mode of evolution was no more than conjectural and largely adjusted to the requirements of the hypothesis. There appears, however, to be no likelihood of reviving the nebular hypothesis or any variation of it. Nevertheless if any alternative hypothesis is to be constructed there is little doubt that more interest would attach to a theory implying the frequent occurrence of solar systems in space than another theory say of the encounter type. In the present paper a further hypothesis for the origin of the solar system satisfying such a requirement is presented.

1. To begin with it may be agreed that the only suitable source for the planetary material must be the stars. It is therefore necessary to consider how material may come to be removed from the main mass of a star where it is normally held by a strong gravitational field. Conceivable processes appear to be limited to the following:—

(i) Collision with another star, (ii) tidal action by another star, (iii) explosion of a single star, (iv) break-up of a single stellar mass due to rotational instability.

Of these four (i) and (ii) involve an external catastrophic mode of origin for the system and therefore need not be further considered here. The third suggestion of explosion of a star does not seem to merit much attention as there would appear to be no physical basis for supposing that there exists any force capable of ejecting the necessary quantities of matter from a star. Admittedly the phenomenon of novæ points to some force capable of disrupting a star, but the nature of the force responsible is at

present unknown and it would not be much gained to attempt to attribute the origin of the planets to an at present unknown or purely speculative mechanism. Process (iv), however, has interesting possibilities and its discussion will be the main topic of this paper.

Before proceeding with this it may be worth while to recall that our theoretical knowledge of the behaviour of rotating fluid masses is confined very largely to the case of uniform density and that consequently there is this restriction on the conclusions reached. This limitation, it will be realised however, is one of degree only, for theoretical conclusions must always refer to idealised conditions and accordingly, in spite of the fact that the theoretical models utilised may not represent the actual situation as closely as could be wished, it will always be permissible in principle to use the theoretical results as a guide to the behaviour of the natural systems with which we are concerned.

2. We shall require the following conclusions obtained from the theory of rotating liquid masses:—

I. The instability of a rotating mass results in its division into two components whose masses are in a ratio greater than 4 : 1 and having hyperbolic relative velocity so that the masses separate to infinity.

II. A stable binary system in which the ratio of the masses of the components is less than 3 : 1 possesses more angular momentum than could be stored in their combined mass (there being no change of density) without rendering it rotationally unstable.

The first conclusion has been discussed in a previous paper *, and it will suffice here to recall that the last stable figure that the mass can have is in the form of an ellipsoid whose axes are in the ratios

$$1{\cdot}886 : 0{\cdot}815 : 0{\cdot}651.$$

Without loss of generality the mass of the ellipsoid can be taken as unity and the units of length and time chosen so that the density is $3/4\pi$ and the constant of gravitation G equal to unity. The angular momentum of the last stable ellipsoid is then 0·389 in these units and the angular velocity at this stage is given by

$$\omega^2/2\pi G\rho = 0{\cdot}142.$$

The second result II and the mass ratio in I are deduced from considerations of the more general form of Roche's problem, which relates to the stability of figure of a mass moving in a circular orbit about another mass, the whole system rotating as a rigid body and both masses being of equal and uniform density. For values of the ratio λ of the two masses from 0·4 to 1 the problem has already been fully investigated by Darwin.† The value $\lambda = 0$ corresponds to Roche's problem. Since the values of λ that turn out to be of chief interest here are less than 0·4 the calculations for $\lambda = 0{\cdot}1$, 0·2 and 0·3 have been carried out to sufficient accuracy to complete the discussion and show the continuous character of the series of configurations as λ varies. The method adopted takes advantage of the smallness of λ and approximations converge rapidly; the procedure is as described in Darwin's original papers. In Darwin's notation the masses of the components are taken as

$$m = \frac{4}{3}\pi\rho \cdot \frac{\lambda}{1+\lambda} \quad \text{and} \quad M = \frac{4}{3}\pi\rho \cdot \frac{1}{1+\lambda}$$

and the distance between their mass centres at the limit of stability is denoted by R. Thus their combined mass would occupy a sphere of unit radius and the ratio of their masses is λ.

* *M.N.*, 98, 646, 1938.

† Darwin, *Scientific Papers*, III, 436–524 (Camb. Univ. Press), 1910.

It was shown by Darwin that to a high degree of accuracy both components can be regarded as ellipsoids so oriented that the longest axes (c and C) are in the line joining the centres of mass, while the two axes of intermediate length (b and B) are in the plane of the orbital motions. In every case it is found, as might be expected, that the smaller mass becomes unstable first as their distance apart decreases. The following table shows the values of the dimensions of the system in the configuration of limiting stability for values of λ from 0 to 1 at intervals of 0·1:—

		$m=\lambda M$			$M=m/\lambda$		
	R	a	b	c	A	B	C
0·0	2·457	·482 / ∞	·511 / ∞	1·000 / ∞	·942	1·030	1·030
·1	2·465	·363	·387	·647	·905	·987	1·018
·2	2·471	·456	·486	·752	·872	·949	1·007
·3	2·477	·517	·553	·807	·842	·916	·997
·4	2·484	·562	·603	·843	·815	·886	·988
·5	2·485	·597	·642	·870	·792	·860	·979
·6	2·490	·627	·674	·888	·772	·836	·969
·7	2·497	·654	·701	·901	·753	·815	·958
·8	2·502	·673	·725	·912	·737	·796	·947
0·9	2·508	·691	·744	·921	·722	·778	·937
1·0	2·514	·708	·762	·927	·708	·762	·927

It is not necessary to consider values of λ greater than 1 as m would then be the larger mass and by interchanging m and M the above table then refers to the instability of M. For $\lambda=0$ the mass m is infinitesimal but its shape, which is what determines its stability, is indicated by giving the ratio of the axes in the limiting form. (It is particularly to be noticed that the analysis assumes equal densities of the components; it is entirely possible that a small mass could render unstable a larger mass of smaller density. This point may be of importance in stellar considerations.)

By means of the foregoing table we can draw up a further table of relevant quantities. Firstly the quantity $R-(c+C)$ will be equal to the shortest distance between the surfaces of the components and is of obvious importance in considering the geometrical possibility or otherwise of the orbital motion. Secondly we may tabulate the length $R+(c+C)$, the overall extent of the binary system, in order to compare it with the size of the critical ellipsoid. The third and fourth quantities in the following table are the angular momentum $\mathbf{h}$ and the angular velocity, in the form $\omega^2/2\pi G\rho$, in the limiting configurations *:—

λ	$R-(c+C)$	$R+(c+C)$	$\mathbf{h}$	$\omega^2/2\pi G\rho$
0·0	1·427	3·487	0·110	0·0449
0·1	·800	4·130	·227	·0444
·2	·712	4·230	·307	·0441
·3	·673	4·281	·363	·0438
·4	·653	4·315	·402	·0435
·5	·636	4·334	·428	·0434
·6	·633	4·347	·446	·0432
·7	·638	4·356	·457	·0428
·8	·643	4·361	·464	·0426
0·9	·650	4·366	·468	·0423
1·0	·660	4·368	·469	·0420
Jacobian ellipsoid	twice longest axis = 3·772		0·389	0·142
$\lambda=0·36$	·660	4·303	0·389	0·0436

* The values of $\omega^2/2\pi G\rho$ for $\lambda=0·4$ to 1 are taken from Jeans, *Problems of Cosmogony*, p. 63 (Camb. Univ. Press), 1917.

The values in the last line, $\lambda = 0{\cdot}36$, have been obtained by interpolation and correspond to a value of the angular momentum equal to that of the critical ellipsoid. The graph of h plotted against λ is shown in fig. 1.

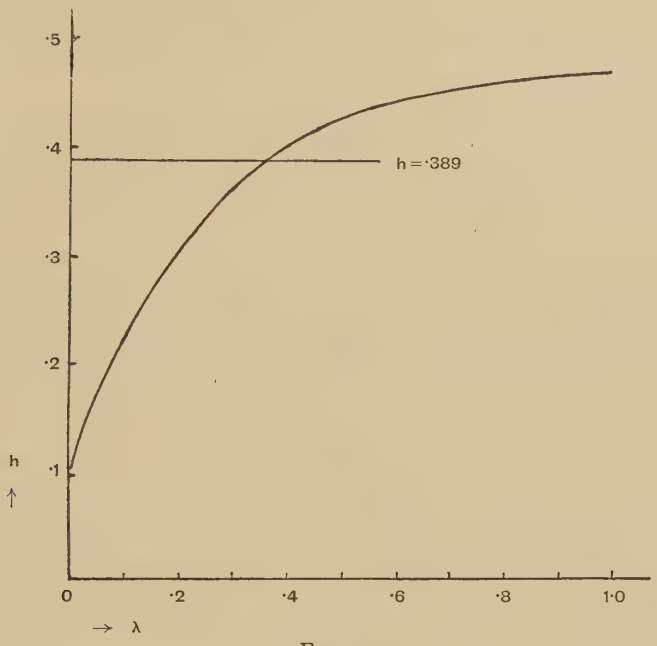

FIG. 1.

It is seen therefore that for all values of λ greater than about 0·36 the binary system will have greater angular momentum than could be stored by their combined mass as a stable body. A larger orbit involving a greater measure of stability for the smaller mass would carry even more angular momentum than the critical configuration.

It is to be noticed that for no value of λ does the limiting position of stability allow the masses to be in contact. The quantity $R-(c+C)$ takes its minimum value for $\lambda = 0{\cdot}6$, about, though it remains practically constant for a very considerable range of values of the mass-ratio.

3. *Effect of Non-uniform Density.*—The type of departure from uniformity to be expected in stellar masses will consist of an increase of density towards the centres of the components. We are interested in determining what conditions have to be satisfied for non-uniform density analogous to those set out in the above tables for the case of uniform density distribution. Suppose therefore that we regard the two components as started in circular motion with their centres at the same distance apart as would give limiting stability if they were of uniform density without change of volume. The angular momentum of this system can be written

$$\mathrm{h} = \{\tfrac{1}{5}m(a^2+b^2)\theta + mMR^2 + \tfrac{1}{5}M(A^2+B^2)\theta\}n,$$

where n is the angular motion in the orbit, while θ is a numerical factor, less than unity, depending on the degree of central condensation and a, b, etc. the axes when the density is uniform. Similarly the angular momentum of the combined mass $m+M=1$ can be written

$$\mathrm{h} = \tfrac{1}{5}(\alpha^2+\beta^2)\theta\omega,$$

where α and β are its semi-axes perpendicular to the rotation axis and ω is the angular velocity. In all cases, except $\lambda = 0$, the term mMR^2n, which is practically independent

of the density distribution, contributes the greatest part of the angular momentum of the binary system, as the following table shows:—

λ	mMR^2n	h (uniform density)
0·1	0·130	0·227
·2	·219	·307
·3	·279	·363
·4	·321	·402
·5	·351	·428
·6	·370	·446
·7	·382	·457
·8	·390	·464
0·9	·395	·468
1·0	·397	·469

On the other hand the angular momentum of the combined mass in limiting form varies considerably, from zero for the Roche model distribution to the value for the Jacobian ellipsoid. Over this range θ varies from 0 to 1, while ω varies along the interval given by

$$0{\cdot}360 \geqslant \frac{\omega^2}{2\pi G\rho} \geqslant 0{\cdot}142,$$

the first number arising from the condition of instability of the Roche model. Thus ω decreases by a factor of about 1·6 while θ changes from 0 to 1. It therefore appears to be valid to infer that central condensation, while hardly affecting the amount of angular momentum of a binary system, in general greatly diminishes the amount of angular momentum that can be stored in a single stable mass. In other words, given a compressible rotating mass in limiting equilibrium, the amount of angular momentum possessed by it would correspond to a binary system of components built on a similar model and having mass-ratio λ *less than* that for uniform density. This conclusion agrees with the result, due to Jeans, that for sufficient compressibility the break-up takes the form of equatorial shedding of matter, which can be regarded as fission with zero mass ratio.

4. *Stability of the Components.*—It is appropriate also to consider as far as possible the figures that the components would have in the case of compressibility and for this purpose they may be supposed to be Roche models. This problem has been discussed by Jeans.* If x, y, z are rotating rectangular axes at the centre of M, z being the axis of rotation and x the line Mm, the total mechanical potential ϕ per unit mass at any point is

$$\phi = \frac{M}{r} + \frac{m}{r'} + \tfrac{1}{2}\omega^2\{(x - mR)^2 + y^2\},$$

where r and r' are the distances from M and m respectively to the point under consideration and R is their distance apart. ($m + M = 1$ and the constant of gravitation $G = 1$.) The free surface that either component could take up is represented by any non-singular member of the family of surfaces obtained by varying ϕ. The critical surfaces will be those possessing equilibrium points. It can easily be seen that such points occur on the line mM joining the mass centres and it will suffice to calculate at what position in this line the equilibrium or neutral point lies for various values of m/M. The position is determined by the condition that $\dfrac{\partial\phi}{\partial x}$ vanishes, or

$$-\frac{M}{x^2} + \frac{m}{(R-x)^2} + \omega^2(x - mR) = 0.$$

* *Loc. cit.*, p. 163.

The quantity ω is related to R by Kepler's equation, so that

$$-\frac{M}{x^2}+\frac{m}{(R-x)^2}+\frac{(M+m)x-mR}{R^3}=0.$$

Taking R as the unit of length and writing $x'=1-x$ and $\lambda=m/M$, this equation is

$$x-1/x^2=\lambda(x'-1/x'^2),$$

and is symmetrical with respect to the two masses as it should be. This equation can be most readily solved for various values of λ by estimating the roots graphically and then obtaining closer approximations. The solutions are as follows:—

$\lambda=0$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
$x=1\cdot000$	0·720	0·659	0·621	0·593	0·571	0·552	0·536	0·523	0·511	0·500

Apart from small values of λ, it is seen that the roots are all close together, the neutral point not being far from the mid-point of *Mm*.

The conditions for stability of figure are now evident. For suppose m to be surrounded by an atmosphere of negligible mass and M to be a point mass with no atmosphere. The atmosphere of m will fill an equipotential surface; but it can do this only so long as its volume is less than that of the critical surface having an equilibrium point. If its atmosphere became larger than this, due to some secular cause, it would begin to flow through the critical point. Moreover, material lost from m would be captured by M, for this is the only way it could move to a position of lower potential energy. Thus the atmosphere of m will be stable provided it does not fill the critical equipotential surface. When both masses possess an atmosphere the condition is the same unless both atmospheres extend beyond the critical point. In this latter event transfer of atmosphere from one mass to another will not in general be possible, as each will have its full possible complement of atmosphere and any extension must now take up an equipotential form having greater volume than the combined volume of the two critical atmospheres of m and M. Thus the system would not remain a double system in appearance, but would at first become a dumb-bell-shaped form slightly larger than the critical figure-of-eight surface. With increasing volume of atmosphere the waist on this figure very rapidly fills in and the shape soon becomes practically spheroidal.

The secular changes supposed capable of increasing the volume of atmosphere may be regarded as occurring physically in either of two ways. In the first place some agency might actually supply atmosphere to m and M, either from within or without. Secondly, each mass may have a constant amount of atmosphere and the point masses be drawn together slowly by some independent secular effect. In either case, or even if the two causes were operating together, it is clear that the course of evolution of the atmosphere would be as described above.

5. We are now in a position to show in what manner a stellar mass may come to be rotationally unstable. Hitherto in discussions of the fission theory of the origin of binary stars it has been an essential assumption that the stellar masses initially possess enough angular momentum eventually to make them unstable. This does not, however, appear to be an altogether justifiable postulate, for it has not been made clear in what way stars could acquire such rotation as would be necessary and furthermore there is no observational evidence that rapidly rotating stars are of such frequent occurrence as the fission theory of double-star systems would seem to require.

Suppose, however, that a pair of stars of comparable mass form a binary system in the condition of limiting equilibrium so that their surfaces are as close together as possible. The angular momentum possessed by this system in virtue solely of the orbital momentum of the two masses would be more than sufficient to cause the combined mass to become rotationally unstable. If now some extraneous secular effect were to operate to bring the stars closer together it is clear that in due course the main bodies

of the stars would be brought within the limits inside which stability is no longer possible. If, for example, the stars consisted of cores of approximately uniform density surrounded by negligible atmospheres, the critical situation would be determined solely by the relative distance of the cores, and the effective surfaces would not be in contact. On the other hand, a considerable degree of central condensation must undoubtedly be regarded as present and a sharp distinction cannot necessarily be drawn between the body and atmosphere of the star. The question then arises as to what will happen when either mass is brought within the limits of stability of the other. If the mass ratio is very large we have the conditions of Roche's problem and it appears to be satisfactory to suppose that the smaller body alone is affected and is completely disrupted. If, however, the bodies are of comparable mass and size the consequences that the instability finally results in would not appear to be capable of being decided by analogy with the Roche problem, for it is obviously unacceptable to suppose that each mass breaks the other into any number of indefinitely small pieces. There is, however, no stable configuration for the system as a dumb-bell-shaped mass, though there appears to be no alternative to supposing that they would begin by uniting into a single mass. If, however, their surfaces came into contact without at first rendering the masses unstable, it is clear that some dissipative action would be brought into play as a result, and the separation, a, would continue to diminish since the total energy, E, would be of the form

$$E = -Mm/2a,$$

and $\frac{dE}{dt}$ would be negative owing to dissipation, so that $\frac{da}{dt}$ would also be negative. Thus whatever the internal density distribution it appears that the two masses will temporarily unite.

The combined mass would, however, retain all the angular momentum originally associated with the binary system and would therefore be rotationally unstable. The situation arrived at is accordingly that met with in discussing the possible outcome of the instability of the Jacobian ellipsoid, except that in the present case the mass may have rather more than the minimum amounts of energy and angular momentum necessary to make it unstable. The mass would therefore divide into two again, the components now having a mass-ratio consistent with the available energy and angular momentum. Also, as has already been shown in a previous paper *, this division would take place in such a way as to cause the two components to separate to infinity.

It is seen that the present hypothesis overcomes the difficulty concerning the source of the rotation, as it is known that many close binaries exist of the type required for the initial conditions of the present mechanism.

6. *Secular Causes of Decreasing Separation.*—As far as the cause of the initial decrease of separation required to start off the foregoing process of coalescence and subsequent break-up is concerned, it would be possible to ascribe it to any one of several possible extraneous effects. The close passage of another star or the chance addition of matter to a binary system are obvious enough examples of occurrences capable of producing the necessary small disturbance. Such explanations, however, would hardly represent an essentially different form of theory from the encounter theories. The type of process that would seem to be most satisfactory would be one to which all or most stars are subject in the course of their general evolution.

Such a process is available in the mechanism of accretion of interstellar mass by stars. The details of this process have been recently discussed † and it has also been shown that the effect of the addition of mass would result in a decrease of the separation of the components of any double star. It is found that the material gathered up by a star from the cosmical cloud must usually have negligible angular momentum and the

* *M.N.*, **98**, 646, 1938.

† F. Hoyle and R. A. Lyttleton, *Proc. Camb. Phil. Soc.*, **35** (1939) and **36** (1940).

evolution therefore takes place in such a way that the angular momentum remains approximately constant. If m_1 and m_2 are the masses, this condition is, with the standard notation,

$$m_1 m_2 (m_1 + m_2)^{-\frac{1}{2}} a^{\frac{1}{2}} (1 - e^2)^{\frac{1}{2}} = \text{constant}.$$

For equal masses this gives

$$a(1 - e^2) \propto m^{-3},$$

where m is the total mass at any time. The continual decrease of $a(1 - e^2)$ at a finite rate will necessarily lead to a collision of the components, for the closest distance is $a(1 - e)$ and this being always less than $a(1 - e^2)$ must also decrease at a finite rate.

7. *Conditions for the Formation of Planets.*—Consider now the evolution of a double star in the light of the foregoing remarks. The system will begin with wide separation and as the mass increases the distance apart will diminish and eventually become comparable with the sizes of the stars themselves. If the components are of similar mass at this stage the system would possess a much greater amount of angular momentum than could be stored in the combined mass as a steady stable form. Further decrease in the separation will eventually render the binary form unstable and the system will temporarily become a single mass though necessarily unstable. This mass will therefore divide into a pair of new components of unequal mass though now with hyperbolic relative velocity. The mass-ratio will depend on the amounts of energy and angular momentum of the system but must be about 3 : 1, at least.

The interesting feature of this course of evolution is that the situation reached when the mass is just on the point of dividing into two is precisely similar to that provided in the collision theory of the formation of planets. The separating masses would produce a filament of material between them in just the same way, while the departure of the two stars to infinity is even more definitely ensured than in the collision theory. The instability of the rotating mass here involved is *ordinary* instability (as opposed to *secular* instability) and its operation does not depend on the presence of dissipative action. This means that the motion must be reversible and the process of fission here described is in this sense seen to be the reversed counterpart of the motion prior to the collision in an encounter between two stars, and accordingly satisfies the requirement of reversibility.

It is now only necessary to regard the double star in the first instance as being associated with the Sun as a companion, the initial system thereby being a triple star comprised of a very close pair (the two parent stars) and a distant body (the Sun). Such systems are by no means rare and accordingly suitable conditions for the formation of planets by this mechanism are likely to be of much more frequent occurrence than for any process relying on an encounter. Moreover, if the theory of evolution of double stars here adopted is correct, it is clear that any star is likely to become first a member of a double system and later a member of a triple system, and so on to a more complex system still. Thus the number of stars capable of providing the requisite initial conditions must be regarded as gradually increasing and this, given sufficient time, to practically any extent.

It would not, however, be permissible to suppose that every such coalescence and subsequent fission of two members of a triple system would necessarily lead to a set of planets associated with the third member; for it is not to be expected that in more than a small proportion of cases would the distant body (the Sun) be in a suitable position for capturing the resulting planets. Nevertheless it can readily be seen that the limitation arising in this way is not very serious—nothing like so restrictive, for example, as the initial requirements in the collision theory. For consider the system when the two components M and m have receded to large distances from one another and suppose that in fig. 2 SG represents the velocity of their centre of mass relative to the Sun and Mm the relative velocity of the masses. If the mass-ratio is taken as 3 : 1

then $3MG = Gm$. Now the planets will have velocity lying between those of the two components. This question has been considered in an earlier paper * for the case of equal masses and for a mass-ratio of 4 : 1; the analysis given applies for any mass-ratio and for the present case of 3 : 1 it is readily found that the velocity of the planets would correspond roughly to that given by the point P in Mm such that $MP : Pm = 3 : 2$. Thus the condition for capture is that while the velocities SM and Sm must be sufficient to enable M and m to escape from the Sun and each other, the velocity SP must be less than the escape velocity from the Sun. Thus there appears to be no very severe restriction in order that the process results successfully, for SG would be of the order of the circular velocity and in general SP would be of the same order as SG.

For definiteness we have in the figure regarded the relative velocity of M and m when at large distance as being of the order of 50 km. per sec., though the conclusions

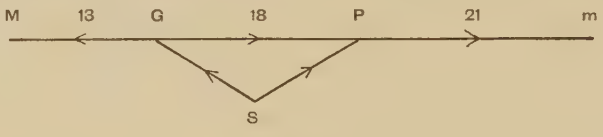

Fig. 2.

reached would obviously apply equally well for velocities of a different order. For fig. 2 concerns only the ratios of the various velocities and would be unaltered in form if the velocity Mm were of any other magnitude, say of the order of 100 km. per sec. In this case, however, the velocity SP would be proportionally increased, but it is evident that this velocity would still be less than the escape velocity provided the encounter took place correspondingly nearer the Sun, the theoretical limit being set only by the maximum value that the escape velocity could have.

On the other hand, apart from the possibility that the process could result in the formation of a planetary system, it is evident that either M or m, or even both, might be retained by the Sun. Again the velocity of the planets relative to the third star might exceed the escape velocity owing to the particular directions in which M and m happened to be projected away from each other. The various possibilities can be represented by means of fig. 2 by adjusting the relative sizes of Mm and GS and the angle between them.

This process of disruption of the combined mass is the same mechanism previously proposed for the origin of the great planets and their satellite systems. In this latter case rotating planets divide by fission into two components and the intermediate material streaming out between them corresponds to the satellites of these planets and the terrestrial planets. It is necessary, however, that the component masses are not both given sufficient velocities by the fission to allow them to escape from the Sun. But in the formation of planets it is essential that the resulting components should escape. There appears to be no difficulty here, however, for the energies and velocities associated with the rotational break-up of a star are very much greater than for a planetary mass of similar mean density, and the residual velocity after the components have separated to great distance would in general be correspondingly higher in the stellar case. Thus for stars similar to the Sun the initial relative velocity of separation of the components would be of the order of 600 km. per sec. and, assuming the motion to be hyperbolic rather than exactly parabolic, the final velocity of separation would probably be of the order of 100 km. per sec. However, for planets like Jupiter or Saturn the relative parabolic velocity at the moment of separation would be of the order of 50 km. per sec., so that, assuming a proportionate amount of energy to be in excess of the parabolic amount, the final relative velocity would be of order 5 km.

* *M.N.*, **98**, 536, 1938.

per sec. and accordingly both the components and the intermediate material could be captured by the Sun.

It may also be noticed that the conditions required for the rotational break-up of a single stellar mass could be reached by means of a collisional encounter between two stars not physically dependent in the initial state of the system. Thus even if the relative velocity of two stars involved in an encounter were insufficient to provide the energy necessary for both their subsequent escape from each other and the removal of the requisite amount of material from their surfaces, it is possible that the relative hyperbolic motion would involve enough angular momentum to render the combined mass unstable.

8. *The Rotations of the Planets.*—The present theory would also be capable of accounting for the rapid rotations of the planets. In the collision theory, it will be recalled, the rotational momentum is ascribed to the transfer of angular momentum by turbulent motion taking place during the collision. Now a double system consisting of two stars similar to the Sun moving in circular orbits about one another with their surfaces in contact would have a rotation period of the order of 6 hours and this is of course of the same order as the rotation period necessary to disrupt a mass of density of the order of unity. By the theorem of the constancy of circulation, it would be expected that in a break-up of a fluid mass under gravitational and rotational forces only, every resulting element of mass would retain momentum giving it a rotation period comparable with that of the original mass.* Subsequent changes of density of any of the elements would affect their rotation, but as far as the great planets are concerned their densities are of the same order of magnitude as that of the Sun and accordingly their initial rotation periods would be of the order of 6 hours. There appear to be indications that the rotation periods of these bodies have been lengthened to their present values.†

9. *Condensation of the Planets.*—It has been objected to the theory of the origin of planets from a collision process that a filament of material of a mass of the order of that of the planets liberated at stellar temperature could not restrain the thermal velocities of its constituent particles by means of its own gravitation. The most detailed attempt to demonstrate this contention is due to Spitzer ‡, who has shown that such a filament *left to itself* would acquire sufficient velocity of expansion to dissipate in a period of the order of an hour at very most. It appears, however, that it is not permissible to regard the filament as conditioned solely by its own gravitational effect and what seems adequate answer to this criticism of the theory is contained in a forthcoming paper. It may, however, serve to complete the present paper if brief reference is made to the matter here.

For simplicity, if the receding stars are regarded as each having a mass equal to that of the Sun, then it is easy to show that in the course of an hour their distance apart would increase from contact (*i.e.* centres at a distance $2R$ apart, R being the radius) to a distance of no more than about $4R$. Thus the filament during this time would be in the combined gravitational field of the stars. At contact of the two stars the energy per unit mass required for escape from the region immediately between them is clearly of order $2GM/R$, while at the end of an hour it would be of order GM/R. This field is obviously far greater than that due to the filament which can have a mass of the order of 10^{-3} that of the stars at most. If it is assumed that the material of the filament is atomic hydrogen, the temperature that would give thermal velocities equal to these escape velocities is of order 2×10^8 degrees. If, however, the average mass of the particles were higher, which it is reasonable to suppose would be the case, the corresponding temperature necessary to cause the filament to dissipate would be proportionally higher. Such a temperature is greatly in excess of that likely to be reached at any part of a star similar to the Sun and, making allowance for the average weight of the

* Jeffreys, *M.N.*, **89**, 636, 1929. † *M.N.*, **98**, 643, 1938. ‡ *Ap. J.*, **90**, 675, 1939.

particles being greater than 1, it turns out that the limiting temperature will be about 10 times higher than is generally recognised as being associated with any ordinary star.

Thus although on the present hypothesis the nature of the disruption implies that the material of the planets must be regarded as coming from very deep within the combined mass and therefore having temperature of order 10^7 degrees, it now appears that this energy would be quite insufficient to enable the filament to expand to any great extent against the gravitational attraction of the parent stars. The material would of course expand outwards as far as it could consistent with its energy before being brought to rest, that is to a state of low temperature, and once this stage was reached the mass of the filament alone could control it after the action of the parent stars had become negligible. The details of this development of the filament are considered more fully elsewhere.

Summary

The possibility of alternative hypotheses for the origin of the solar system is discussed and it appears that the removal of material from a star would be most likely to occur through the effect of rotational instability. A single mass could come to obtain sufficient angular momentum to cause its instability as a result of the components of a double star combining into a single body. The decrease of separation required to instigate this process is a consequence of the general course of evolution of double stars. When the combined mass undergoes fission the two new components separate with hyperbolic velocity and accordingly their initial configuration and motion reproduces the same situation met with in the collision theory of the origin of the planets. The process of planet formation is thereafter the same as in the earlier theory.

In the present theory the Sun is therefore to be regarded as in the first place being the distant companion of a close double star whose components first merge into a single body and then this mass breaks up owing to rotational instability into a pair of stars and these produce the planetary filament as they separate from one another. Such triple systems are by no means rare in space and the formation of planets by this process may therefore be of very much more frequent occurrence than formation by any mechanism of the encounter type.

ON THE ACCRETION THEORY OF STELLAR EVOLUTION

F. Hoyle and R. A. Lyttleton

(Received 1941 March 8)

1. *Introduction.*—During the last two years the writers have advanced a new theory of the rate of addition of mass to a star by the process of accretion of hydrogen.* The formula arrived at for the rate of change of mass when the motion has become steady may be written as

$$\frac{\partial M_A}{\partial t} = 18\gamma^2 M^2 \rho / \bar{v}^3, \qquad (1)$$

where ρ is the density of the cloud in the neighbourhood of the star, M the mass of the star, γ the constant of gravitation and $\bar{v}$ the relative velocity of the star and cloud appropriately averaged to allow for the motion of the star in the galaxy. This formula involves a rate of increase of mass greater for many stars by a factor of the order of 10^6 than the rate arrived at on the basis of earlier formulæ. However, the advantages to theoretical astronomy of this large rate of accretion are manifold and a number of them have been explained by us in some detail.* In spite of these successes the derivation of this formula and its application to certain problems have been subjected to some criticism. Thus in two recent papers † discussing our work conclusions have been advanced by Atkinson that differ so widely from the views we have advocated that some statement of the position in regard to this problem as it now stands seems desirable. The present paper is accordingly concerned with setting out the main hypotheses upon which the derivation and application of formula (1) depend. It will be seen from the discussion that follows that the divergence between Atkinson's point of view and our own depends upon how far these hypotheses can be maintained and it will be shown in this paper that the assumptions adopted lie entirely within the range of valid argument and that our investigations are in accordance with accepted principles.

2. *The Derivation of the Accretion Formula.*—If a particle of mass m describes a hyperbolic orbit about a star and has velocity v relative to the star when moving at infinite distance, then the total energy of the particle at any stage in its motion is $\frac{1}{2}mv^2$, provided any internal energy possessed by the particle is neglected. (The mass of the particle is supposed infinitesimal compared with that of the star.) Thus the total energy remains positive and is constituted entirely of a mass motion of the centre of gravity of the particle together with its negative gravitational energy. It follows therefore that the only means by which such a single particle could be captured is by direct collision with the body of the star. If the cosmical cloud could be considered as being composed of independent particles, so that collisions could be neglected in the accretion process, then only those particles that directly collided with the star could be captured. It was in this way that the older formulæ for accretion were calculated.‡ In our first paper on this question it was shown, however, that the assumption that collisions may be neglected cannot be justified even as a first approximation in dealing with this problem and that despite the low density of the cloud collisions produced by the gravitational action of the star may introduce important dissipative processes. Thus the mass motion of gaseous material as it streams past a star does not vary in the

* *Proc. Camb. Phil. Soc.*, **35**, 405, 592, 1939; and **36**, 325 and 424, 1940.

† (1) *Proc. Camb. Phil. Soc.*, **36**, 313, 1940; and (2) *M.N.*, **100**, 500, 1940.

‡ Jeffreys, *M.N.*, **77**, 98, 1916; and Eddington, *Internal Constitution of the Stars*, p. 391, 1930.

same way as in the case of an assembly of independent particles and there occurs a transference of energy from organised mass motion of the material to random thermal energy. In addition it was found possible to estimate quantitatively the loss of energy of mass motion and to begin with we made the assumption that only the mass motion is of consequence in determining whether the material escapes from the gravitational field of the star. Although the energy possessed by the material consisted of three parts, viz. (i) the mass motion of the material plus its negative gravitational potential energy, (ii) the thermal energy produced by additional collisions due to the gravitational field of the star, (iii) the thermal energy possessed by the material when at infinity, together with any thermal energy developed on approaching the star by the absorption of radiation—it was assumed that only the portion (i) of this energy was of importance for assisting the escape of the material and the problem of how the portions (ii) and (iii) were concerned was then left over for future consideration.* The formula (1) had been deduced on this assumption and although the method was necessarily of an approximate nature the investigation could hardly introduce an error of much more than a factor of 2. Atkinson's first paper is almost entirely concerned with pointing out that the contributions (ii) and (iii) cannot be neglected and argues that since the sum of (i), (ii) and (iii) is positive the material must therefore of necessity evaporate away to infinity. In any case this objection does not seem valid, for in any random distribution of particles a finite fraction of them must have velocity less than an assigned escape velocity; thus, if $\bar{c}$ is the average velocity in a Maxwellian distribution of particles, the rate of effusion † into the surrounding empty space is the same as if the material moved outwards with velocity $\frac{1}{4}\bar{c}$. Accordingly $\bar{c}$ might be above the escape velocity and yet a considerable fraction of the material not escape. In other words, the motion once having become unorganised, it must remain so, and there can be no possibility of more than a fraction of the particles escaping.

But quite apart from this, the method we adopted would be completely justified if the energy contributions (ii) and (iii) are removed in some way, as, for instance, by the direct emission of radiation. Thus if some process of emission could be found that results in the loss of the energy contributions (ii) and (iii) in a time interval short compared with the time intervals that are significant in the accretion process, then the formula (i) could be accepted with considerable confidence. In his paper (1) Atkinson makes it clear that his criticisms of our initial treatment are based on the assumption that such radiative processes do not arise. Before the publication of either of Atkinson's papers we had already considered this aspect of the problem and, by making two assumptions about the structure of the hydrogen cloud, had been able to show that a highly effective emission process was forthcoming. These assumptions are:

(*a*) That the cosmical cloud in its regions of highest density contains an appreciable proportion of hydrogen molecules—10 per cent. by mass would suffice.

(*b*) That the cloud is not everywhere uniformly distributed, but possesses local irregularities.

The thermal energy (ii) and (iii) is then found to be dissipated by the quadrupole vibrational-rotational transitions of the $1'\Sigma$ state of the hydrogen molecules. A detailed account of the operation of this process has now been published.‡

The assumption (*b*) means that if a portion of the orbit of a star in the galaxy, say 1000 parsecs of its length, is considered and the average density of the cloud taken along this section of the track is ρ, then the star is not to be regarded as passing always through material of density ρ but as passing through comparatively short regions of much higher density separated by regions of much lower density. For example, the

* *Proc. Camb. Phil. Soc.*, **35**, 597, lines 24–35, 1939.

† Jeans, *Dynamical Theory of Gases*, p. 121 (Camb. Univ. Press, 1925).

‡ *Proc. Camb. Phil. Soc.*, **36**, 424, 1940.

star might spend $\frac{1}{10}$ of the time passing through material of density 10ρ and be in a region practically devoid of cloud for the remaining $\frac{9}{10}$ of the time. This assumption does not so far require any particular linear dimensions for the irregularities and they could even be small compared with the average interstellar distance. Thus the assumption concerns only the local distribution of the cloud and has no reference to the question of large scale distribution that will be considered in the next section of this paper. Accordingly it means that for two stars with widely separated orbits, no information is given as to whether the average densities are equal along the two orbits, the hypothesis merely requires the cloud to be patchily distributed along both orbits. The necessity for this assumption appears to arise in particular for the case of the stars of classes O and B, for it is found that hydrogen molecules can only reach the capture radius of these stars (apparently an essential requirement if accretion is to take place at the full rate given by the formula) if the density lies above a certain limit that depends on the surface temperature of the star and its total luminosity. For stars of very high surface temperature and great luminosity this lower limit to the density would seem to exceed the highest value that could reasonably be assigned to the density of the cloud averaged along the whole orbit of the star. The assumption of irregular distribution overcomes this difficulty.

3. *The Application of the Formula.*—The most immediate application of the formula (1) concerns the resolution of the difficulty of the short lifetimes of massive stars and it was the consideration of this problem that led to the view that the energy of the stars must be constantly resupplied from without. By supposing that the cloud is constituted mainly of hydrogen the lifetimes of the stars are not limited to the time required for the transmutation of the amount of hydrogen present in the stars at any given time. If $\partial M_T/\partial t$ represents the mass of hydrogen in grams transmuted within the star per second, then

$$\frac{\partial M_T}{\partial t} = 125\frac{L}{c^2} = 1{\cdot}4 \times 10^{-19} \, . \, L,$$

where L is the luminosity in ergs per second. If the star consists of a mass αM of hydrogen and a mass $(1-\alpha)M$ not hydrogen, the condition that the proportion of hydrogen is kept up to this value α by a rate of accretion $\partial M_A/\partial t$ is

$$\frac{\frac{\partial M_A}{\partial t} - 125\frac{L}{c^2}}{\alpha} \geqslant \frac{124\frac{L}{c^2}}{1-\alpha} \quad \text{or} \quad \frac{\partial M_A}{\partial t} \geqslant \frac{L}{c^2}\frac{125-\alpha}{1-\alpha}.$$

Thus if α corresponds to 25 per cent. hydrogen by mass the required condition may be written

$$18\gamma^2 M^2 \rho/\bar{v}^3 \geqslant 1{\cdot}85 \times 10^{-19} \quad L. \qquad (2)$$

If $\bar{v}$ is *estimated* from the observed peculiar radial velocities, then for any particular star of known M and L the minimum value of ρ satisfying (2) can be determined. This gives the order of magnitude of the minimum value of the average density of material along the orbit of the star. For stars of large mass the minimum value of ρ is about 2×10^{-21} gm. per c.c.* In paper (2) Atkinson has arrived at almost the same value as this.

A rough estimate for the average value of ρ throughout the galaxy can be obtained from Kepler's third law applied to the motion of the Sun. There is an uncertainty by a factor of order 10 in the result arrived at, however, due to the combined effect of uncertainties in such quantities as the linear velocity of the Sun in the galaxy, the distance of the Sun from the galactic centre, the general shape of the galaxy and the question of the existence of spiral arms. A discussion of this has been given in a recent

* *Proc. Camb. Phil. Soc.*, **36**, 325, 1940.

paper *, where it was shown that a value of about $2 \cdot 10^{-22}$ gm. per c.c. would represent a reasonable upper limit for the average density of interstellar material lying within the orbit of the Sun. Atkinson in his paper (2) also considers an upper limit of this order, the actual value adopted being 10^{-22} gm. per c.c., and he concludes that no reasonable value of ρ can satisfy (2) for highly luminous stars. This conclusion does not seem to be warranted, however, for it assumes that the average value of ρ along the orbit of the particular star under consideration is of the same order as the average density throughout the whole of the galaxy lying within 10,000 parsecs of the centre. There is no dynamical reason why this should be the case and what observational evidence there is does not confirm it. It seems more reasonable to assume that highly luminous stars exist only in exceptional regions of the galaxy where the average value of ρ along their orbits is higher by a factor of order 10 than the average value for the whole galaxy. Thus in our view the whole, or at any rate a large part, of the orbit of a highly luminous star is to be regarded as lying in an exceptional region of the galaxy. As a simple model of the galaxy that illustrates how this could happen we may picture the cosmical cloud as being highly concentrated towards the galactic plane, so that in regions within 100 parsecs of this plane the density is of order 10^{-21} gm. per c.c. The distribution may of course involve other departures from uniformity in addition to this, such as the existence of spiral arms.

The requirements of the accretion theory as it is at present formulated may be summarised in the following three hypotheses:—

(*a*) That the cosmical cloud in its regions of highest density contains an appreciable proportion of hydrogen molecules—10 per cent. would suffice.

(*b*) That the cosmical cloud is not everywhere evenly distributed but possesses local small irregularities.

(*c*) That the cosmical cloud is irregularly distributed also on a large scale, and in particular it is strongly concentrated towards the galactic plane, where the density rises to a value of order 10^{-21} gm. per c.c.

4. *The Hypotheses* (*a*), (*b*) *and* (*c*).—In the first place, so far as investigations of the consequences of the accretion process have gone, none of these hypotheses introduces any astrophysical difficulty and no contradictions have been encountered, while the number of facts explained far exceeds the number of assumptions made. This in itself is regarded in science as sufficient justification for the introduction and adoption of a hypothesis. Moreover, in the present case, the denial of any of the hypotheses (*a*), (*b*) and (*c*) would automatically require the introduction of an alternative new hypothesis to take its place. This means that in making these assumptions the position of the theory is not weakened in respect of the principle that has been termed "the economy of postulates." Thus, for example, if the hypothesis (*b*) is denied in favour of a uniform distribution of the cloud, then (*b*) is merely replaced by the hypothesis that the density of the cloud is constant. (This is not to say that the hypotheses (*a*), (*b*) and (*c*) may not ultimately be shown to be consequences of other hypotheses of wider significance.) The case for the acceptance of (*a*), (*b*) and (*c*) then depends entirely on how much can be predicted and explained (correlated) by theory working from these assumptions. In the present problem the fact that the accretion theory promises to meet successfully the difficulties concerning the lifetime of the stars and the origin and evolution of binary systems, and in addition provides many other satisfactory results, indicates that these assumptions can be regarded with the completest confidence that can be accorded to any hypothesis not directly verifiable by observation. This aspect of the accretion theory has been dealt with in our earlier papers, so that although this argument forms the strongest evidence in favour of hypotheses (*a*), (*b*) and (*c*) it will not be necessary to give here any recapitulation of this work.

There are a number of further arguments that afford interesting confirmation of

* *Proc. Camb. Phil. Soc.*, **36**, 325, 1940.

the conclusions arrived at in these earlier papers. For instance, the adoption of hypothesis (*a*) requires an appreciable proportion of the cosmical cloud to be in molecular form. In order to make an estimate of the probability of forming hydrogen molecules from separate atoms the cross-section for this process is required and this unfortunately is not at present available either from theory or experiment. On the other hand, it is of the greatest interest to find that direct observational confirmation of the occurrence of vibrational-rotational transitions in the molecules *CH* and *CN* in the cosmical cloud has recently been found by Adams * and McKellar.† Thus although our prediction that the regions of high density in the cloud are at low temperature, compared with the well-known estimate due to Eddington, might otherwise have been regarded with some reserve, McKellar's results afford strong observational support for this conclusion.

The assumption (*c*) requiring a density of the order of 10^{-21} gm. per c.c. in regions close to the galactic plane will probably be regarded as open to some question and we shall therefore give further considerations that lend additional support to this postulate. In the first place, this assumption yields the immediate result that the most massive and luminous stars may be expected to be strongly concentrated to the galactic plane. This agrees with observation and thereby obtains a result not previously accounted for by theoretical astronomy. Secondly, it is reasonable to suppose from dynamical considerations that in a rotating mass of gas the density will rise to maximum value on the plane of symmetry determined by its rotation axis.

The possible suggestion that the value 10^{-21} gm. per c.c. represents too high a density near the galactic plane is an objection that, if it applies to all, can apply only to the galaxy itself. For, by means of observation of a number of extra-galactic nebulæ of the spiral type, Jeans ‡ has obtained values of the order of 10^{-21} gm. per c.c. for the average central density of these bodies. Accordingly objections such as those urged by Atkinson in (2) could not in any case apply to the stars situated in these external galaxies. Since, however, the evolutionary property of the stars, so far as have been ascertained, seem to be much the same for our galaxy as for the extra-galactic nebulæ, this affords further support for the hypothesis (*c*). It would of course be possible to argue that the values of the velocities of the stars relative to the cloud differ systematically between the galaxy and the external nebulæ, but the adoption of this as an *ad hoc* assumption to avoid the resolution of an acknowledged theoretical difficulty would be not only extremely artificial but also contrary to scientific practice.

5. A number of other points of Atkinson's paper (2) seem to require comment. In the first place, although it is not explicitly stated, the general tenor of the paper gives the impression that the author believes that the accretion process is merely in the course of being considered as a possible way out of certain astrophysical difficulties. This in itself seems to be nothing more than a possibly unrecognised continuance of the attitude that characterised former researches on stellar evolution in which the origin and early history of the stars were vaguely passed over by ascribing them, somewhat arbitrarily, to condensation in parent nebulæ.§ In our view, however, the stars must not (*a priori*) be considered as completely autonomous bodies. Moreover, the accretion process is not by any means to be regarded merely as a further speculative hypothesis designed to extend the age of the stars, but is a process that must in any case take place on the scale that we have proposed, as will be shown in a moment. The only possible alternative would be in supposing that the stars have not condensed at all but have always been compact; such an assumption, however, achieving nothing more than its own statement, is so unplausible as to be outside the scope of serious discussion.

Returning now to the previous point, it can easily be seen from general considera-

* *Harvard Announcement Card*, 526, 1940.
† *P.A.S.P.*, **52**, 187, 1940; and *Mt. Wilson Annual Report*, p. 18, 1941.
‡ *Astronomy and Cosmogony*, p. 331 (1929 ed.).
§ For example, see *Internal Constitution of the Stars*, p. 17.

tions that the rate of increase of mass of the Sun, say, at some time in its history must have been of the same order as that given by our formula, at the very least. For, taking the present mass of the Sun as 2×10^{33} grams and the age of the universe as 10^{11} years, the average rate of increase of mass during this period is 2×10^{22} gm. per year or 6×10^{14} gm. per sec., while, according to our formula, the present rate of accretion must be of the order of 10^{14} gm. per sec., which establishes the result. Evidently then, unless the age of the universe is very much greater than 10^{11} years, other factors must have affected the accretion of the Sun. For example, the Sun may have possessed a companion body of comparable mass for a considerable part of its existence, and this could to some extent increase the accretion rate. More important still, owing to its application to the stars generally, is the possibility that the velocity relative to the cloud may in effect be less than the peculiar velocity. (See also below.) Furthermore, it cannot yet be decided whether the density of the cloud can be regarded as not changing with time and it is not inconceivable that it was different in the past.

As Atkinson has pointed out, an effective capture radius S may always be utilised irrespective of the actual mechanism of accretion and the accretion rate is then $\pi S^2 \rho v$. If we equate this to 10^{14} gm. per sec. for the Sun, a condition between the capture radius and density is thereby obtained, viz.:

$$S^2 \rho = 1{\cdot}6 \times 10^7.$$

It follows that if the density were of order 10^{-23} gm. per c.c. the capture radius would be of the order of 100 astronomical units; while if the capture radius were of the order of the radius of the Sun, as the other extreme, the density must be of the order of 10^{-15} gm. per c.c. Unless it can be shown how the capture radius could have such a high value, the first of these results rules out the value generally adopted for the density, while the second result shows that the capture process can have no direct relation to the radius of the star.

The foregoing remarks refer to increase of mass only, without regard to the composition of the matter added, and as such apply to all stars. For the very luminous and massive stars, however, the most pressing difficulty presented by them involves periods of time of a different order of magnitude and the need was for an additional source of energy. In solving this problem by supplying hydrogen from without, it is incidental that this involves also a rapid increase of mass, but it is this feature that, without further assumption, solves the problem of the evolution of binary systems. Thus two difficulties are overcome by a single step.

6. In discussing the dynamical evolution of binary systems Atkinson criticises our work from two standpoints, neither of which can be upheld. The first concerns the incorrect conclusion that the equation (2) can never be satisfied, and an instance is considered in which the accretion is so small that the rate of increase of mass by accretion is less than the rate of decrease of mass by radiation, *i.e.*

$$\frac{\partial M_A}{\partial t} < \frac{1}{125} \cdot \frac{\partial M_T}{\partial t}.$$

In such a case Atkinson then argues that no dynamical evolution can take place no matter how long a time-scale is taken ("whatever the time-scale and whatever the energy source"). This conclusion is incorrect. The actual state of affairs is that, given sufficient time, *no matter how slowly accretion takes place*, it can increase the mass of a star to any extent, whereas radiation, *no matter how fast it takes place*, can alter the mass of a star by no more than one part in 125 of the final mass arrived at by accretion. Atkinson's error arises on account of the unnoticed assumption that if the above inequality holds at a particular time then it holds for all time, whereas the inequality must always be reversed over some period in the case of a time-scale greater than 10^{10} years.

Secondly, Atkinson writes that it is "only at the price of extending the time-scale to 10^{10} years or more" that the requirements of binary evolution can be overcome by accretion. It is correct that the accretion theory does indicate an age for the stellar universe (in the galaxy) of order 5×10^{10} years, a value in excess of the usual estimates by a factor of at least 5. There is no mention in Atkinson's paper, however, of what "price" it is that would have to be paid in so extending the time-scale. Our calculation * of a lower limit of 3×10^{10} years for the age of the companion of Sirius, of such obvious importance in the present connection, is passed over without reference. There appears to be no cogent reason why the time-scale should not be extended in accordance with the indications of the accretion theory of stellar evolution.

Although the long time-scale as advocated by Jeans of 10^{12} or 10^{13} years has fallen out of favour the main idea behind Jeans' work remains as important as before—namely, that sufficient time must have elapsed to produce the extensive dynamical evolution that observations force us to regard the stars as having undergone. If the time-scale becomes so cramped that there is insufficient time for widespread dynamical evolution the "price" seems to be in the opposite sense to that used by Atkinson. Such questions as the dynamical evolution of binary systems as a result of accretion and the lifetime of the companion of Sirius provide strong evidence for a time-scale in excess of 10^{10} years.

7. It is further maintained by Atkinson that if the rate of accretion were given by a formula of the type proposed by us, an observable distinction should exist sharply separating stars of small velocity from those of slightly higher velocity (p. 509, lines 22–26), and hence, since this is not observed to be the case, that accretion cannot be given by our formula. In answering this objection several important points arise. In the first place, in making this suggestion Atkinson appears to have ignored our demonstration † that the velocity of a star relative to the cloud may vary by a factor of at least two as it pursues its orbit in the galaxy and hence that even if the velocity at any instant were known, its use for v in the formula might introduce an error of a factor of about 3 in the average accretion rate. For, if we denote the average rate of accretion by $\bar{A}$, our analysis shows that in the course of describing an orbit in the galaxy the rate varies between about $\frac{1}{3}\bar{A}$ and $\frac{5}{3}\bar{A}$, corresponding to the maximum and minimum values of the velocity. This is, of course, on the assumption of uniform density all along the orbit. Thus there is no reason to suppose that two stars having peculiar velocities of equal magnitude at a given time and situated in the same region of the galaxy would necessarily have equal average rates of accretion.

This alone would be sufficient answer to Atkinson's claim, but the question of the value of v to be adopted leads to other considerations of interest. Even on the assumption that the motion of the cloud is strictly in circular orbits, it is not certain that the peculiar velocity of a star, which depends on its motion relative to other stars, coincides with its velocity relative to the cloud, though there seems to be no reason to expect these velocities to be greatly different. However, the velocity appears cubed in the formula and quite a small error in estimating v could therefore lead to a more considerable error in the calculated accretion. Such an uncertainty may very well be involved in the velocities in any case, for it has to be remembered that the so-called K-term, which refers to stars of low velocity in particular and for these stars is of the same order as the peculiar velocities themselves, is not yet satisfactorily explained.

For stars of small mass having sufficient energy provision apart from future accretion, for periods of order 10^9 years or longer, the calculation of their total accretion would require a knowledge of v over such intervals and it would not be permissible to suppose that the value of the peculiar velocity would remain at its present value throughout. The origin of the peculiar velocities of the stars has not yet been fully explained; it is hoped to examine this question in subsequent work.

* *Proc. Camb. Phil. Soc.*, 35, 603, 1939. † *Proc. Camb. Phil. Soc.*, 35, 600, 1939.

There is the further point that if the cloud is not regarded as uniformly distributed, the denser regions would have their separate motions and might themselves be expected to possess velocities differing appreciably from the circular velocity. Bearing all these factors in mind, it is clear that in using the peculiar velocity of the star for the value of v in the formula, the result so obtained is not likely to give more than an order of magnitude estimate of the accretion rate. The uncertainties in the parameters so involved are unavoidable at the present time, but this cannot in any way be regarded as a defect of the theory itself.

We may in passing refer to a criticism of a minor nature that has been advanced concerning the application of the accretion theory to the question of changes of the solar radiation. It was shown * that a change of 10 per cent. in the Sun's radiation would occur for $v = 10$ km. per sec. and $\rho = 10^{-18}$ gm. per c.c., and it has been pointed out that the value 10^{-18} gm. per c.c. may not actually be attained in any nebulous cloud. It is clear at once, however, that if the maximum density available should be only 10^{-19} gm. per c.c., say, the same change in the Sun's radiation would be produced for a relative velocity of about 5 km. per sec. for the star and cloud. That such relative velocities may occur at times is confirmed by the measures of radial velocity in certain nebulæ.†

8. In conclusion we may review the general position and development of the accretion theory. Before its advent physical theory had progressed sufficiently to make practically certain that the process of transmutation of hydrogen must provide almost the whole of stellar energy, while the available astronomical evidence, particularly from double stars, led us to the view that a further potential source of energy must be introduced from outside the stars, either continually or intermittently replenishing the hydrogen in the star. Once this idea was clearly grasped there was little difficulty in conceiving the broader requirements of the accretion theory; the details, however, were not so readily forthcoming. The existence of interstellar matter in gaseous form, at any rate in certain regions of the galaxy, had already been established. However, the presence of such a cloud as this could provide no help in the problem. The reason that this is so is worth mentioning as it does not seem to have been appreciated. In the first place, the chief constituent of the cloud was regarded as calcium, a number of similar elements such as sodium possibly being present also, but the addition of such matter to a star and the consequent increase of mass could not extend the life of a star, but would if anything shorten it. Thus, in stating that it had been generally recognised by astronomers that the problem of the energy supply of bright stars could be solved by addition of matter to them, Dr. Atkinson draws attention to what must have been a widespread misunderstanding of the true nature of the problem. In Eddington's account of the process of the accretion of mass‡, however, this mistake is certainly not made and there is no suggestion of the process leading to any extension of the life of stars; this is further evidenced by the fact that the accreted material is taken as having atomic weight 10. In the event of material other than hydrogen being supplied to the star the less is accreted the better as far as the lifetimes are concerned. It is true that at the time of publication of Eddington's work on this question some doubt yet remained as to whether the source of energy were due to the complete annihilation of matter or to transmutation of hydrogen. So long as the speculation involved in the former of these possibilities was retained then of course accretion of mass would extend the life-times, but once the source of energy is ascribed to the second process then mere accretion of mass is useless, and accretion of hydrogen only is effective.

In our own discussions of the problem we simply postulate the hydrogen cloud and proceed to consider what conditions the cloud would have to satisfy. Unfortunately

* *Proc. Camb. Phil. Soc.*, **35**, 412, 1939.

† Russell, Dugan and Stewart, vol. ii, 829.

‡ *I.C.S.*, pp. 391 and 392.

we were encountered at the outset with what seemed to be an insuperable obstacle to any progress with the problem. We refer, of course, to the question of the motion of the gas in the gravitational field of the star. As there was no hydrodynamical precedent to guide us, the only possibility seemed to lie in getting some simple picture of the process that would enable the order of magnitude of the rate of accretion to be calculated. Any such solution must necessarily involve some assumption additional to those that would lead to the rigorous solution of the problem and the justification of such a procedure would then rest on how much the solution turned out to be capable of explaining. After considerable attention had been given to the problem it was decided that a formula of the type we eventually proposed would be most likely to give the best estimate. The most striking feature of this formula was the enormous rate of increase of mass that it involved, even if the density of the cloud were no higher than the estimate of 2×10^{-23} gm. per c.c. given for the calcium, etc., cloud.

Now in so solving the problem of the source of energy of the bright stars, we had invoked a hypothesis that also solved the question of the dynamical evolution of binary stars. The situation with regard to this second problem was to some extent similar to that of the former, for here the attention of astronomers was largely engaged with the fission theory of the origin of binary systems. Thus at a single blow this new process resolved the chief difficulties of these hitherto widely separated problems and thereby unified the dynamical and physical evolution of the stars. Moreover, the process proposed cannot be regarded as speculative in the sense, for example, that the hypothesis of the complete annihilation of matter was speculative. The latter hypothesis at the time of its consideration was not only a process so far unobserved to occur, but even with its postulated properties it did not by any means fully and consistently explain the phenomena to which it referred. On the other hand, in the accretion process there is involved no question of a mechanism unknown to science, while as far as considerations of the consequences of the process have proceeded no contradictions have yet appeared.

The situation is, therefore, that so far as investigations have progressed we have developed within the limitations fixed by the inherent mathematical difficulties of the problems involved a consistent theory of stellar evolution. If this theory, including its hypotheses, were completely confirmed by the present observations of the entities involved, such as the density and structure of the cloud, the velocities, luminosities, compositions, etc., of the stars, then the work would have achieved nothing more than showing the logical interrelation of such quantities. But in reality our work is at present of greater potential value than this for it shows that (on the basis of the present accepted physical theory) a consistent theory can only be built if the parameters associated with certain factors take fairly definite values. For instance, the theory indicates that the density of hydrogen in space (which *must* be assigned some value), averages at least 10^{-22} gm. per c.c. throughout the main part of the galaxy, while rising to a value above 10^{-21} gm. per c.c. in some regions. Furthermore, as has been seen, there is nothing in the available evidence bearing on this question to show that such a requirement cannot be fulfilled. If, however, future investigations should show that these conditions cannot be accepted then it would be necessary to modify the theory without sacrificing its consistency, or, if this were not possible, to construct a new theory. It will be seen from these remarks that in criticising our work by maintaining that the required density is impossibly high (or higher than at present seems permissible) Atkinson is attacking the theory on a point where it is not exposed to any attack until definite evidence can be cited against the density adopted.*

9. The position may be illustrated very clearly by recalling some analogous questions. In attempting to account for the anomalous motion of Uranus, the postulate of additional material in the solar system was made by J. C. Adams, who then

* *Observatory*, **63**, 39, February 1940; and *Proc. Camb. Phil. Soc.*, **36**, 325, 1940.

endeavoured to calculate, as best as could be managed in such a difficult problem, the value that the density would have to take. In order to simplify the treatment as much as possible it was supposed that the whole matter could be regarded as concentrated in a sphere moving at a certain distance from the Sun, the density elsewhere being zero. On this basis the question of the density distribution resolved itself into a determination of the co-ordinates of the centre of the sphere. A critic of this entirely legitimate procedure could apparently have objected that the density taken by Adams was far beyond any that then seemed permissible, while it would also have been possible to criticise the analysis employed. Quite possibly these objections were actually advanced at the time, but Adams's investigations on the problem would have been equally justifiable whether Neptune had consequently been discovered or not.

In the same way, as a possible explanation of the peculiarity of the motion of Mercury outstanding from its Newtonian orbit, a distribution of material was again postulated and the required density calculated. In this case, however, for the theory to be logically consistent (after accepting the usual gravitational equations, etc.) the density was found to be quite discordant with more direct evidence from observations, the amount of matter theoretically required far exceeding that which could possibly be present. Accordingly the proposed explanation had to be discarded.

These examples show what the correct attitude towards the accretion theory of stellar evolution should be and that much of Atkinson's criticisms has no valid basis. The real need at present in this problem is for trustworthy observations leading to information of the density distribution and velocities of the stars relative to the cloud. At present, in regard to stellar evolution, the choice is between the consistent theory based on the idea of accretion and no theory at all. In making a decision between these two, before it is argued that the problem may be conditioned by some effect as yet unrecognised or unknown to science and that the agreements achieved by the accretion theory are due to pure coincidence, it must first be recalled that such an argument would dispose of each and every scientific theory yet invented. It is not within the claim of any theory to show that the same conclusions could or could not follow from other hypotheses at present unknown; the proper purpose of a theory is to construct a model that successfully predicts and correlates phenomena within what are called the errors of observation.

Summary

The discussion of accretion given in a recent paper by Atkinson and the criticism of our work involved in it are considered in some detail. It is shown that most of the arguments therein advanced had been dealt with by us before the publication of Atkinson's article, while several errors of method have led to a number of other invalid criticisms being put forward. It is shown that when these errors are set right the difficulties met with by their author are removed.

VARIATIONS IN THE PERIODS OF CERTAIN ECLIPSING BINARIES

H. Horrocks, M.A.

(Received 1940 December 11)

1. *Introduction.*—Light elements, by means of which times of minima of eclipsing binaries are expressed, may require the inclusion of non-linear terms. This circumstance may arise from such causes as a rotation of the apse due to non-spherical form of close components, or to the presence of a third body in the system. In such a case, the times of minima exhibit periodic oscillations about the times given by a linear law embodying the mean period of light variation. If such an eclipsing binary, however, is observed over an insufficient fraction of the period of these superposed oscillations, the phenomenon may appear as a quadratic term in the light elements, the linear part of which is unlikely to represent the true mean period of light variation. Continuation of the series of observations may reveal the true character of the deviation of the times of minima from a linear law—a situation which has not infrequently arisen.

In the compilation and reduction of the observational material contained in the present paper, we also had in mind the problem of testing whether the observational data would suffice to determine the possible existence or probable absence of a measurable systematic progressive change of period amongst eclipsing binaries satisfying certain criteria of selection, after the exclusion of those whose variations of period seemed to be predominantly oscillatory. Some interest attaches to this problem from a purely observational point of view of determining, if possible, what the existing phenomena are.

The work out of which the present investigation arose, led to the limitation of the systematic search for observational data to eclipsing binaries whose periods are between 4 days and 20 days. A further limitation was imposed by requiring the masses and periods to satisfy a certain criterion the details of which will not be described here. Of the stars thus selected, all have been examined whose periods are given in Schneller's *Katalog und Eph. Veränd. Sterne für 1937* to six or more decimals.*

If successive periods of light variation are subject to an increment ΔP, and this may be regarded as constant over the interval covered by the observations with which we are concerned, times of corresponding minima may be expressed in the form

$$t = (P_0 + \tfrac{1}{2}\Delta P)E + \tfrac{1}{2}\Delta P.E^2.$$

Here E is the number of complete periods since the epoch $t = 0$, at which the period just completed was P_0.

An examination of the times of minima of an eclipsing binary for evidence concerning progressive variation of period requires the determination of quadratic light elements, in which the coefficient of E^2 is $\frac{1}{2}\Delta P$, which might be written $\frac{1}{2}P\dot{P}$ when ΔP is small. Eclipsing binaries for which this coefficient can best be determined are those of short period. They offer scope for observing a greater number of minima, and greater accuracy is to be expected in the timing of their minima through the steepness of their light-curves before and after minimum light. Moreover, since the probable errors of determinations of $\frac{1}{2}\Delta P$ for two binaries with a similar distribution of equally accurate observations over the same interval of time would be inversely proportional to the squares of their extreme ranges of E, the probable errors of their resulting values of $\dot{P}$ would be proportional to P.

For a discussion of such material in relation to the possibility of the existence of a

* *Kleinere Veröff. Univ. Sternw. Berlin Bab.*, nr. 16, 1936.

systematic progressive variation of period affecting eclipsing binaries generally, the values of $\dot{P}$ derived from coefficients of E^2 would be combined into suitably weighted means, preferably in groups according to period. Spurious values of $\dot{P}$ arise, however, when stars with variations of period due to the presence of oscillatory terms in the light elements are observed over an insufficient interval. It would thus be necessary for the stars to be sufficiently numerous, and for the spurious contributions to $\dot{P}$ from this cause not to be excessive compared with any such systematic variation of period, in order that the effects of oscillatory terms not established as such may be sufficiently reduced or partly eliminated in forming the weighted means. In actual fact the eclipsing binaries suitable for discussion, within the scope of this investigation, are found to be too few, and it appears probable that any such systematic variation of period, if existent, is too small compared with spurious values that may arise as above for this procedure to be possible, at least at present.

Two of the stars included in this paper give strong evidence that an oscillatory term is required in the light elements. For stars for which solutions with quadratic light elements have been made, the expression for times of minima has been put in the form

$$O-C=x+y\frac{E-E_0}{1000}+z\left\{\frac{E-E_0}{1000}\right\}^2.$$

Here $(O-C)$ is the residual from preliminary linear light elements, and E_0 is a suitably chosen origin of E for the solution. The residuals were weighted and grouped into three, four or six normals. In assigning weighted mean residuals to weighted mean values of E, an approximation is involved when there is sensible departure from linear light elements. For the data under discussion the consequent error is of little importance. Moreover, if the distribution of observations for a given star were nearly the same in each group, these errors would affect chiefly the determination of x, which is the term of least physical interest. Even for the two systems for which a quadratic term large compared with its probable error has been found in the present reductions, the necessary corrections are small and have not been applied.

In determining the coefficients x, y, z of the quadratic correction to the preliminary linear light elements, the observational equations were combined by the method of least squares when six normal residuals were formed. In cases where four normal residuals were formed, E_0 was taken as the mean of the E's of the first and last normal residuals. The difference of the first and last equations was formed to obtain y, and their mean was then compared with the mean of the second and third equations to obtain x and z.

In examining the evidence for or against a measurable change of period, it is of some importance to assign probable errors to the normal residuals formed. We may then obtain the above coefficients together with their probable errors. This information would be of considerable importance for a discussion as to whether a small progressive variation of period of a systematic character is to be found affecting eclipsing binaries generally.

Probable errors or mean square errors are given by some authors, and others give material in a form from which these—as far as internal agreement of a single series of observation is concerned—can readily be derived. When data for several stars are available by the same author, the probable errors from internal evidence which are below average have been somewhat increased. In some cases a normal minimum is given without a probable error and without data for individual minima, but a light-curve is given as well as individual observations of magnitude, and perhaps a statement of the probable error of a magnitude observation. In such cases the probable error, from internal evidence, of a single determination of a time of mid-minimum may be at once derived. It has been assumed that the normal minimum is based on most of the observations suitable for deriving the time of mid-eclipse from the magnitude observations and light-curve given.

As, in general, material from different sources is being combined, the above probable errors based on internal agreement have been increased, the proportional increase being greater of course for the smaller probable errors. Their finally assigned values are therefore to some extent arbitrary, and we have no certain means of testing their accuracy. It may be remarked, however, that of the two stars for which six normals were formed, RS CVn gives final residuals in good agreement with what would be expected on the basis of the assigned probable errors, whilst TT Lyræ gives final residuals much smaller than might have been anticipated. As regards the other stars, normal minima from different sources that have been combined into single normals in the present discussion have been separately compared with the final solutions given in Table II, and there are about as many larger than their assigned probable errors as there are smaller, a few large residuals being perhaps excessive in magnitude. It would therefore appear that, on the average, the system of assigned probable errors is satisfactory.

The next three sections summarise the data collected. Within the adopted range of search, it is hoped that the material was reasonably complete at the time at which it was compiled.

2. *Eclipsing Binaries whose Periods are Variable, and an Oscillatory Character is probably established for the Variation.*—(1) *SY Cygni.* Nijland * gives minima during the years 1910 to 1915 together with the formula

$$242\,0001{\cdot}571 + 6^{d}{\cdot}005716E,$$

the residuals with respect to which are exhibited in fig. 1.

Five minima in 1906 and 1907 are given by Graff.† Nijland ‡ remarks on the discordance between these minima and his formula quoted above, and suggests that it is due to the differences between their concluded light-curves and the unsymmetrical distribution of observations between the ascending and descending branches. We have reduced Graff's magnitude observations, suitable for determinations of times of minima, to the zero and scale of Nijland's magnitudes (judged by the magnitudes given for maximum and minimum) by a small linear correction, and derived times of minima using Nijland's light-curve. The times of mid-minimum for the same eclipse, derived from observations at different parts of the descending or ascending branches of the light-curve, show a systematic run. The question arises whether this is due to systematic differences between their magnitude scales such that the one is not transformed into the other by a linear correction, or due to a real change of light-curve between the two series arising from a rotation of the apse combined with the eccentricity of the orbit and the inclination of its plane to the line of sight.§ In fig. 1 we plot the residuals, from the foregoing formula, of the times of the five minima as given by Graff and as computed above.

Pickering ‖ gives results for some early minima, from 1890 to 1894, which have been reduced to the form of residuals from the above elements. As the data were scanty on which his approximate light-curve could be based, considerable error in these minima may be expected. Using the photographic magnitude $10^{m}{\cdot}83$ at maximum from Pickering's data, and assuming a photographic range of $2^{m}{\cdot}5$, a linear correction has been applied to give the magnitudes with Nijland's zero and scale. One observation is too near maximum magnitude to give a trustworthy time for mid-minimum, and has been omitted from this reduction. A second is near to maximum light and so will give a time of mid-minimum subject to greater uncertainty than the average. A third falling $0^{m}{\cdot}5$ from minimum light, on our assumption, will be to some extent sensitive to any error in the magnitude range we have assumed. The four residuals computed as above are plotted in fig. 1 as well as the residuals derived from the times given by Pickering.

* *A.N.*, **228**, 177, 1926, nr. 5459. † *Mitt. Hamb. Sternw.*, nr. 11, 37, 1907.
‡ *A.N.*, **203**, 395, 1917, nr. 4871.
§ A brief discussion of the results in relation to apsidal rotation is given later.
‖ *A.N.*, **152**, 89, 1900, nr. 3630.

Ceraski * gives the times of mid-exposure of four plates on which the variable was invisible. These times give the following residuals:—

$E =$	-1089	-1086	-929	-916
$O-C =$	$+0^d.078$	$-0^d.083$	$+0^d.164$	$+0^d.098$

The range of the residuals, $0^d.247$, is just about the duration of minimum light as given by Nijland. The residual of mid-minimum should therefore be fairly near the mean of

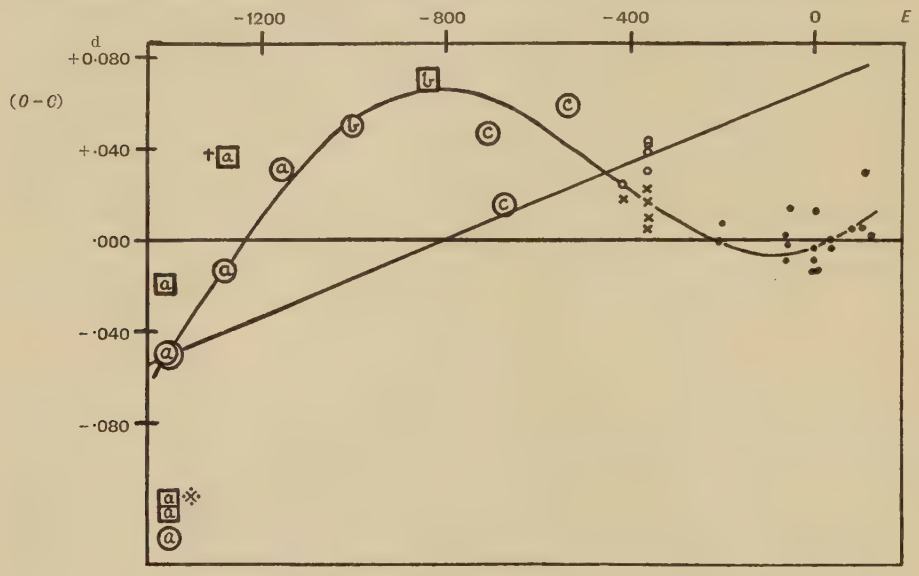

FIG. 1.—*Residuals of the observed minima of SY Cygni from the formula*

242 0001·571 $+6^d.005716E$.

(a) *Pickering.*

[a] *Pickering minima reduced with Nijland's light-curve.*

* *Sensitive to assumed minimum magnitude.*

† *Near maximum light.*

(b) *Ceraski ; variable invisible.*

[b] *Blajko ; observation interpreted with Nijland's light-curve.*

(c) *Parkhurst.*

○ *Graff.*

× *Graff minima reduced with Nijland's light-curve.*

● *Nijland.*

the extreme residuals, which happens to be not greatly different from the mean of the four. We will thus assign the residual $+0^d.05$ to the epoch -1000, and it is perhaps probable that the error does not exceed $0^d.03$.

He also describes an observation made by Blajko when the star was fainter than normal and increasing in light. By Nijland's light-curve we estimate that the time of mid-minimum would be about 2^h earlier than that suggested by Ceraski. On this basis we obtain the residual $+0^d.07$ at $E = -832$, for which the error is perhaps not greater than $0^d.03$. These residuals are shown in fig. 1.

Parkhurst † gives 14 estimates of magnitude, of which 11 appear to be at maximum light and 3 appear to be at minimum light. As determining the times of mid-minimum, the latter may thus be in error by as much as about $0^d.12$ on the basis of Nijland's deter-

* *A.N.*, 151, 224, 1900, nr. 3614.

† *A.J.*, 24, 160, 1905, no. 571.

mination of the duration of constant minimum. Their residuals, however, are in remarkable agreement and are plotted in fig. 1.

In spite of the considerable uncertainties attaching to some of the data, an oscillatory term appears to be required in the light elements. Its period is perhaps about 2000 orbital periods, or 30 years, and its amplitude $0^d.07$. The mean period corresponding to this would be about $6^d.00580$. Such a variation, represented by a sine term, is indicated in the figure by the ordinate-difference between the curve and the straight line which represents the linear part of the correction required by the preliminary light elements. Its evaluation depends considerably on early uncertain data, but a few later minima determined with modern precision would be invaluable for a determination of the character of the light elements.

(2) *W Delphini.* Technically this system is outside the scope of the present discussion, the period being given to only 4 decimals in *Kat. und Eph. Veränd. Sterne* (1937). The period, however, is variable, and as the system satisfies our criteria in other respects it has been included, the available series of observations being extensive.

Data for times of minima from 1896 to 1927 have been collected by Graff *, who gives the formula

$$241\ 8048{\cdot}679 + 4^d.806E - 0^d.006\left(\frac{E}{1000}\right)^2,$$

which does not, however, represent the early observations. We have reduced the data given by Graff to the form of residuals from the linear formula

$$241\ 8048{\cdot}679 + 4^d.8060E$$

and exhibit the results graphically in fig. 2. Later data are given by Graff † (a minimum in 1930) and by Piotrowski ‡ (two normal minima, 1933 and 1934). By comparison with the above linear formula, these give

Source	E	$O-C$ (days)
Graff	+1686	−·038
Piotrowski	+1919	−·033
,,	+2002	−·030

which are also exhibited in fig. 2.

An oscillatory term is strongly indicated. The straight line in the figure represents a tentatively suggested correction to the preliminary linear light elements, to determine the mean period. Representing the superposed oscillation by a sine term, its period appears to be about 3600 orbital periods (about 50 years) and its amplitude about $0^d.046$, whilst it vanishes at about $E = -600$ and $E = +1200$. The mean period in this case would be 4·806029 days, and the constant term in the formula for times of minima would be 241 8048·631.

3. *Eclipsing Binaries whose Periods are Variable, but the Variation is not yet established as Oscillatory.*—(1) *RS Canum Venaticorum.* Extensive data have been collected by Sitterly §, who gives residuals of minima and of normal minima on the basis of the formula

$$242\ 2811{\cdot}666 + 4^d.797851E.$$

He derives corrected light elements, including a harmonic term, which represent the observations from 1901 to 1928, but two early Harvard minima (1892 and 1893) are markedly discordant. A later normal minimum by Himpel || reduced on the basis of the above formula gives

$$E = +1055, \qquad O-C = +0^d.125 \pm 0^d.007 \quad (? \text{ m.s.e.}).$$

* *Atti Pont. Acc. Nuovi Lincei*, 83, 104, 1930. † *B.Z.*, 12, 62, 1930.
‡ *Acta Astr.*, ser. c, 2, 61 and 77, 1934. § *Contr. Princeton Univ. Obs.*, No. 11, part 2, 1930.
|| *A.N.*, 261, 231, 1936, nr. 6253.

This, together with the two early Harvard minima, suggests that much of the departure from a linear expression in E is due either to a periodic term of much longer period than that given, or to a quadratic term. We have solved for a correction to the above elements,

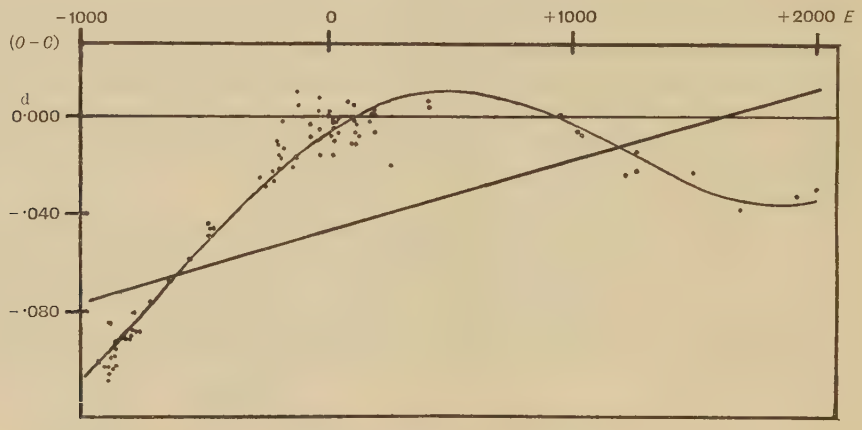

Fig. 2.—*Residuals of the observed minima of W Delphini from the formula* 241 8048·679 + 4^{d}·8060E.

adding a term in E^2. The material has been divided into six groups as follows, giving a moderately symmetrical distribution of the normal minima:—

E =	−2197 to −1290	−1232 to −1004	−991 to −546	−535 to −165	−4 to +160	+508 to +1055

(2) *SW Cygni.* This system, whose period is not constant, has been discussed by Parenago *, who collected data into eight normal minima with a range of 2447 periods. His result has been incorporated in Table II.

(3) *RT Lacertæ.* Normal minima from observations by Luizet and by Nijland as well as his own are collected by Wachmann †, who gives their residuals compared with the formula

$$242\ 1913{\cdot}483 + 5^{d}{\cdot}073921E,$$

as well as with a new determination of the elements. To these we have added results from an early series of observations by Enebo‡, and a later series by Himpel.§ As regards the former, his zero epoch is not central in his series of observations. We have adopted as the normal minimum for this series of observations, that computed by his elements for an epoch 25 periods later than his zero epoch. Compared with the above formula, these additional normal minima give:

Source	E	$O-C$
		d
Enebo	− 682	−0·015
Himpel	+1204	+0·088 ± 0·006 (? m.s.e.)

These normals have been combined into three groups: the two by Luizet with that of Enebo, the three by Nijland, and Wachmann's normal with Himpel's. In each group the means have been formed assigning equal weight. A significant quadratic term is found, the mean period during the interval covered by the observations being 5^{d}·073980, differing little from the period given by Wachmann.

* *R.A.J.*, 8, 229, 1931.
‡ *Beob. Veränd. Sterne*, 4, 46, 1910.
† *A.N.*, 255, 341, 1935, nr. 6115–16.
§ *A.N.*, 261, 233, 1936, nr. 6253.

4. *Eclipsing Binaries for which a Variation of Period is not established.*—(1) *TW Andromedæ.* Dugan * has collected data concerning minima between 1909 and 1914 as well as normal minima in 1925 and 1930, giving residuals by comparison with the formula

$$242\ 0051{\cdot}620 + 4^{d}{\cdot}122745E.$$

Later normal minima are given by Miczaika † and by Lause.‡ Rejecting two discordant early minima, and omitting Gadomski's normal minimum for which the residual is less accordant with the remaining data than is an earlier determination of his based on fewer observations, we have the following normal minima §, with residuals on the basis of the above formula and with assigned probable errors :—

E	$O-C$	Assigned *p.e.*
	d	d
− 182	+0·0004	±0·0020
+1442	·000	± ·004
+1669	− ·003	± ·007
+1881	− ·0056	± ·005
+1954	− ·0075	± ·004

For the later epochs the data are based on rather scanty material, though they agree remarkably well in indicating a shorter period at that epoch than the average period over the whole series. Unless, however, excessive values have been adopted for the assigned probable errors, it remains an open question whether there is any evidence of a change of period in this system, though Gadomski's normal minima lend support to the view that the period is decreasing.

From the four recent minima the correction to the basic formula is

$$-0^{d}{\cdot}004 \pm 0^{d}{\cdot}002 + 10^{-6}[-14 \pm 11][E - 1700].$$

Comparing this with the early normal minimum, the average period between $E = -182$ and $E = +1700$ differs from the basic formula by 10^{-6} $[-2 \pm 1{\cdot}5]$. From the change of period in 940 periods we obtain

$$\dot{P} = 10^{-9}[-3 \pm 3].$$

(2) *SX Draconis.* Nijland ‖ gives the formula

$$242\ 1743{\cdot}643 + 5^{d}{\cdot}169258E,$$

based on minima which we have divided into the following two groups for the formation of normals: $E - 224$ to -20 and $E - 3$ to $+312$.

Van Biesbroeck ¶ and Enebo ** give earlier normal minima from which we derive the following residuals from the above formula:—

Source	E	$O-C$
		d
Van Biesbroeck	−577	−0·023
Enebo	−498	− ·030

These epochs are 20 and 100 periods respectively later than the zero epochs of these authors' elements, in order to reduce them approximately to the mean epochs of their observations. They have been combined with weights in the ratio 3 : 1.

* *Contr. Princeton Univ. Obs.*, No. 14, 1933.

† *A.N.*, 258, 410, 1936, nr. 6192. ‡ *A.N.*, 260, 289, 1936, nr. 6233.

§ Shapley's normal minimum has been assigned to $E = +90$, as an approximate mid-epoch in the series of observations on which it is based.

‖ *A.N.*, 234, 97, 1928, nr. 5598.

¶ *Ann. Obs. Roy. Belg.*, II, 13, 240, 1914. ** *Beob. Veränd. Sterne*, VII, 45, 1913.

(3) *TT Lyræ.* A series of photographic observations extending from 1890 to 1913 is given by Harwood *, who tabulates the observed magnitudes and times of exposure, and the phases of the observations referred to a given formula for the times of minima. Nijland † gives minima from 1911 to 1915, and Lause ‡ gives a normal minimum based on 7 minima in 1935. The formula

$$241\,9680{\cdot}648 + 5^{d}{\cdot}243719E,$$

in which a correction has been applied to Nijland's period, well represents all these observations. The residuals of the observed minima compared with this formula, arranged according to the adopted groups, are given in Table I. The results for the

Table I

Residuals of the Observed Minima of TT Lyræ from the Formula

$241\,9680{\cdot}648 + 5^{d}{\cdot}243719E$

(Unit of $O-C = 0^{d}{\cdot}001$)

Harwood		Harwood		Nijland and Harwood *		Nijland	
E	*O−C*	*E*	*O−C*	*E*	*O−C*	*E*	*O−C*
−1539	− 2	−276	+11	−83	+ 1	+ 68	+ 7
−1326	− 6	−276	+ 3	−79	+ 6	+ 72	+ 4
		−234	− 8	−71	− 2	+103	− 7
		−222	−10	−45	+23	+107	− 7
− 919	−10	−210	+ 3	−30	−26	+115	−16
− 919	− 4	−210	+ 8	−26	− 8	+122	+ 2
− 911	+ 3	−202	− 5	−25	+23	+123	+ 2
− 911	0	−144	+ 9	−16	+22	+126	+10
− 768	+ 9	−144	0	−13	− 9	+130	0
− 768	− 6			−12	+ 5	+134	+16
− 706	+ 9			− 9	+ 1	+138	+17
− 652	− 2			− 2	− 4	+151	− 4
− 633	− 7			0	+10	+172	+ 4
− 563	− 9			+15	+11	+180	− 5
− 559	+11			+37 *	− 6	+196	+ 8
− 547	− 8			+41 *	− 3	+204	+ 4
− 497	+24			+60	−11	+205	− 4
				+62	−18	+208	+ 4
				+65 *	+13	+213	− 3
				+66	+ 9		
						Lause (normal)	
						+1591	+ 2·7

Harwood minima are based on a reduction of her observations which give magnitudes between 10·00 and 11·20, using her tabulated phases and light-curve.

(4) *UU Ophiuchi.* Light elements

$$241\,8833{\cdot}505 + 4^{d}{\cdot}396766E,$$

* *H.A.*, 84, No. 2, 37, 1916.
† *A.N.*, 239, 249, 1930, nr. 5727.
‡ *A.N.*, 259, 189, 1936, nr. 6204.

and residuals of 14 minima extending over a range of 729 periods, are given by Waterfield.* Four groups with moderately uniform distribution have been formed by grouping 2, 6, 4 and 2 minima.

(5) *RV Telescopii.* Light elements

$$241\ 8641{\cdot}1876 + 8^{d}{\cdot}328078E,$$

and residuals of observations of 18 minima extending over 866 periods, are given by Waterfield.† Four groups with moderately uniform distribution have been formed by grouping 6, 4, 6 and 2 minima.

(6) *RS Vulpeculæ.* Normal minima by Maggini ‡, Nijland §, Baker ||, Dugan ¶ and Mergentaler ** have been compared with Nijland's formula

$$242\ 0606{\cdot}638 + 4^{d}{\cdot}477664E,$$

and the first three have been combined into a single normal. With estimates of their probable errors from internal evidence, these three are as follows:—

Source	E	$O-C$	*p.e.*
		d	d
Maggini	−188	−0·038	±0·0025
Nijland	0	·000	± ·0020
Baker	+ 74	− ·008	± ·0012

These have been combined with weights in the ratios $\frac{1}{3} : \frac{1}{2} : 1$.

Some attempt has been made to refer the normal minima by Maggini, Baker, and by Dugan to the mean epochs, approximately, of their observed minima by adding 25, 300 and 500 of their concluded periods to their zero epochs respectively. From internal evidence, the probable error of the normal minimum from Dugan's observations is estimated at $\pm 0^{d}{\cdot}0012$, whilst Mergentaler gives $\pm 0^{d}{\cdot}010$ (? m.s.e.).

It may be remarked concerning this star that Dugan, from photometric observations, found a secondary minimum, moderately well defined, an hour and a half more than half a period after primary minimum. On the other hand Baker, from photographic observations, did not find a secondary minimum though his solution of the primary minimum for orbital elements requires a shallow secondary minimum two hours less than half a period after primary minimum. Nijland found no evidence of a secondary minimum, though unlike Dugan and Baker he found asymmetry in the light-curve of primary minimum.

5. *Discussion of Stars for which an Oscillatory Term has not been established.*—Of the stars examined, the foregoing are all for which sufficient data were found for inclusion in the present investigation. Those suitable for solution including a quadratic term, from normal minima suitably distributed over more than 600 periods, are collected in Table II, which gives the preliminary linear light elements, the coefficients of the quadratic corrections with their probable errors, and the data on which the solutions are based. All the solutions are due to the present writer except that for SW Cygni, which is quoted from Parenago who remarks that an observation subsequent to those reduced by him is not represented by the formula as given.

The first three binaries are those for which a variation of period appears to be well established. For the remaining five stars, the coefficients of E^2 are all less than $1\frac{1}{2}$ of their probable errors.

From an examination of these solutions, it appears that the best series of observations at present available suffice to determine the coefficient of E^2 with a probable error of the order $\pm 2^{d} \times 10^{-9}$ or $3^{d} \times 10^{-9}$ for binaries with periods of about 5 days. Two of the

* *H.B.*, 852, 1927. † *H.B.*, 851, 1927.
‡ *A.N.*, 200, 54, 1915, nr. 4779.
§ *A.N.*, 244, 209, 1932, nr. 5843.
|| *Publ. Univ. Missouri Obs.*, 32, 1921.
¶ *Contr. Princeton Univ. Obs.*, No. 6, 1924.
** *Acta Astr.*, ser. c, 2, 64, 1934.

TABLE II

Correction of Preliminary Linear Light Elements of Eclipsing Binaries in the Form

$$O-C=x+y\frac{E-E_0}{1000}+z\left(\frac{E-E_0}{1000}\right)^2$$

Star	Preliminary Linear Light Elements, and E_0	Normal Residuals (unit $0^d.001$) E $O-C$ *p.e.*	Coefficients (unit $0^d.001$) x and *p.e.* y and *p.e.* z and *p.e.*
RS CVn	242 2811·666 +4·797851 E $E_0=-200$	−1609 +12 ± 6 −1129 − 2 ± 7 − 681 − 9 ± 7 − 389 −20·5 ± 4 + 58 + 6·5 ± 2 + 782 +89 ± 5	− 8·9 ± 1·9 +49·5 ± 3·1 +48·8 ± 3·3
SW Cyg	241 8440·758 +4·5728184 E $E_0=0$	Discussion by Parenago	0 ± 2 0·0 ± 2·2 −70·6 ± 2·2
RT Lac	242 1913·483 +5·073921 E $E_0=+228$	− 590 − 2 ± 8 + 11 − 1 ± 5 +1046 +95 ± 6	+ 9 ± 6 +59 ± 6 +56 ±12
SX Dra	242 1743·643 +5·169258 E $E_0=-216$	− 557 −25 ± 5 − 128 − 0·5 ± 5 + 124 + 1 ± 4	− 3 ± 6 +38 ± 9 −76 ±55
TT Lyr	241 9680·648 +5·243719 E $E_0=+50$	−1432 − 4 ± 7 − 719 + 1 ± 3·5 − 213 + 1 ± 4 − 3 + 2 ± 3·5 + 146 + 1·5 ± 3·5 +1591 + 2·5 ± 4	+ 1·9 ± 2·0 + 1·7 ± 2·3 − 0·9 ± 2·0
UU Oph	241 8833·505 +4·396766 E $E_0=+665$	+ 336 − 4·5 ±11 + 531 + 1·5 ± 7 + 715 − 3·5 ± 8 + 994 − 6·5 ±11	− 1 ± 6 − 3 ±24 −45 ±97
RV Tel	241 8641·1876 +8·328078 E $E_0=+93$	− 236 − 2·5 ± 5 + 10 − 2 ± 6 + 225 + 2·5 ± 5 + 422 + 2 ± 8	0 ± 4 + 7 ±14 − 3 ±65
RS Vul	242 0606·638 +4·477664 E $E_0=+622$	+ 6 −11 ± 3 + 500 −14 ± 4 +1237 − 9 ±12	−14 ± 4 + 2 ±10 +10 ±18

three systems affording this degree of accuracy show such large and widely different values of the coefficient that the variation of period in those cases must clearly be attributed to causes peculiar to each. Over the intervals covered by the observations, the departures of their times of minima from linear laws are comparable in order of magnitude with the ranges suggested for oscillatory terms in SY Cygni and W Delphini, and it is to be anticipated that they will prove to be of a similar nature. The observed deviations

of the minima of RT Lacertæ from a linear law, though not so well determined, are of comparable order of magnitude and are to be expected to be similar in character also.

For an examination of the results in relation to the question whether there exists a progressive change of period of a systematic nature, affecting eclipsing binaries generally, there remain only three binaries whose rates of change of period are determined with probable errors which are not so large that they must at once be excluded from further discussion. These, with the values of $\dot{P}$ derived for them, are as follows:—

Star	$\dot{P}$
TW And	$(-3 \pm 3)10^{-9}$
TT Lyr	$(-0{\cdot}35 \pm 0{\cdot}75)10^{-9}$
RS Vul	$(+4{\cdot}5 \pm 8{\cdot}0)10^{-9}$

The material comprising the present investigation does not yield an adequate number of well-determined values of $\dot{P}$ to furnish observational evidence concerning the possibility of the existence of such a systematic variation of periods, beyond the suggestion that if any such phenomenon exists, it is so small that a large body of observational material, yielding results at least comparable in reliability with the best of the present determinations, will be required to reveal it.

6. *Discussion in Relation to the Possible Interpretation of the Data for some Stars in Terms of Apsidal Rotation.—SY Cygni.* The data collected for times of minima indicate an oscillatory period. In so far as they suffice for the determination of light elements, we have found them to be adequately represented by the adoption of a mean period of $6^d{\cdot}00580$ and a superposed oscillation expressed as a sine term of amplitude $0^d{\cdot}07$ and period 30 years.

With this period, the interval between the mean epochs of Graff's and Nijland's observations is about a fifth of the period of the oscillation. If this is due to a uniform advance of periastron, Graff's observations were made shortly after eclipses occurred at apastron. The observed minima would then be a little early, and the observed period would have just passed its minimum. Nijland's observations, made when the major axis of the orbit was next approximately normal to the line of sight, show the maximum difference between the times of observed minima (early) and mean minima, the observed period agreeing with the mean period.

In the case of a binary with a uniform advance of periastron and constant eccentricity, not too large, for which the eclipses are central, if E is counted from the eclipse at periastron, times of minima are given approximately by *

$$t = t_0 + EP + \frac{eP}{\pi} \sin 2\pi \frac{P}{P'} E,$$

where P is the mean (orbital) period and P' is the period of the revolution of the apse. If the minima of SY Cygni have an oscillatory term of the magnitude suggested, this formula would give 0·037 for the orbital eccentricity.

W Delphini. If after due correction for the mean period of light variation, the oscillatory term remaining in the times of minima for this binary has the magnitude suggested, viz. an amplitude of $0^d{\cdot}046$, the above formula would give 0·030 for the orbital eccentricity.

TT Lyræ. Shapley † finds no evidence from Nijland's observations for or against eccentricity of the orbit, and finds the data insufficient to say whether the components are spherical or spheroidal. M. B. Shapley ‡ finds "positive evidence of a light-variation due to the ellipsoidal form of the components" from Harwood's observations. They

* See, *e.g.*, Pannekoek, *Untersuchungen über den Lichtwechsel Algols*, p. 28, 1902.

† *Contr. Princeton Univ. Obs.*, No. 3, 1915.

‡ *Ap. J.*, 46, 56, 1917.

do not reveal any orbital eccentricity, and for her solution she adopts a circular orbit. Using her values $a_b = 0{\cdot}175$ and $a_f = 0{\cdot}249$ for the radii of the components, the ratio of the orbital period to the period of revolution of the apse, computed by the formula given by Russell *, is

$$\frac{P}{P'} = \frac{6{\cdot}4}{10^4} k_b + \frac{1{\cdot}7}{100} k_f,$$

where k_b and k_f are factors depending on the internal constitution of the stars. It must be noticed that, as a consequence of the great difference of brightness between the two sides of the faint component found by M. B. Shapley, a difficulty arises if the mass ratio is to be derived from the mass-luminosity law. The value 0·42 has been adopted. An error in this would affect chiefly the coefficient of k_f.

For Y Cygni, Russell and Dugan † found $k = 0{\cdot}034$ by assuming that k has the same value for both components. If k for each component of TT Lyræ ‡ is of this order of magnitude, P/P' is of the order 1/1700. Hence in such a case the observations here discussed would cover more than one complete revolution of the apse, and the absence of any marked evidence of an oscillatory term would indicate a negligible value of the orbital eccentricity.

Summary

Data for times of minimum light have been collected from various sources for a selection of eclipsing binaries. They have been examined with a view to testing the constancy or otherwise of their periods. Oscillatory variations are suggested for the periods of SY Cygni and W Delphini.

For other binaries, for which adequate material has been collected, quadratic light elements have been derived, together with the probable errors of the coefficients on the basis of a system of probable errors assigned to the data, which seems to be satisfactory. A fairly well-determined large coefficient of E^2 has been found for RS Canum Venaticorum, SW Cygni and RT Lacertæ. It has been suggested that the variations of period in these cases are probably due to oscillatory terms whose periods are too long for the observations at present available to reveal their true character.

Of the remaining stars for which quadratic light elements have been derived, only TT Lyræ and RS Vulpeculæ lead to determinations of rates of change of period with probable errors which are not excessive. For TW Andromedæ a value for $\dot{P}$ has been derived by a different method, with a moderate probable error.

The results obtained have been briefly discussed from the point of view of a search for evidence whether or not the periods of eclipsing binaries show a systematic progressive variation which is not too small to escape detection. Some of the stars have also been discussed in relation to apsidal rotation.

* *M.N.*, **88**, 641, 1928. † *M.N.*, **91**, 212, 1930.

‡ It may be remarked that available data for the masses make Y Cygni considerably more massive than TT Lyræ.